Non-Commutative Iwasawa Theory With (,)-Local Conditions Over Distribution Algebras

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Abstract

In this thesis we formulate a natural non-commutative Iwasawa Main Conjecture for motives which fulfil the Dabrowski-Panchishkin condition on the level of (φ, Γ)-modules. The basic framework we employ is still Fukaya-Kato's but we work systematically over Schneider-Teitelbaum's distribution algebras of compact p-adic Lie groups instead of Iwasawa algebras. This allows us to consider as local conditions not just subrepresentations of the p-adic realisation which fulfil the Dabrowski-Panchishkin conditions but also sub-(φ, Γ)-modules which fulfil the analogous Dabrowski-Panchishkin conditions. We then combine this with Pottharst's Selmer complexes and a generalisation of Nakamura's Local Epsilon Conjecture for (φ, Γ)-modules to conjecturally define p-adic L-functions. We prove that the validity of our main conjecture for these p-adic L-functions follows from the validity of Fukaya-Kato's Equivariant Tamagawa Number Conjecture and our generalisation of Nakamura's Local Epsilon Conjecture. Moreover we are also able to compute the values of these p-adic L-functions at motivic points.

Our formalism allows us, for example, to unify the GL_2-main conjecture of elliptic curves which have either ordinary or supersingular reduction at p. In addition, we can use our formalism to give a new, and very natural, interpretation of Pollack's $\pm$-construction in the context of supersingular elliptic curves and we are hopeful that this new interpretation will in the future lead to the construction of natural non-commutative generalizations.

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0 Introduction

0.1 Overview

Our goal is to improve the understanding of the (non-commutative) Iwasawa theory of motives that do not necessarily fulfil the Dabrowski-Panchishkin condition at p. One of the most important examples of such a motive is that which arises from an elliptic curve with good supersingular reduction at a prime p.

Let us briefly review the situation of an elliptic curve over $\mathbb{Q}$ with good ordinary reduction at p. Let T_pE be the p-adic Tate module associated with E. The elliptic curve has ordinary reduction at p if T_pE is reducible as a representation of the local Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. Then the only non-trivial subrepresentation is the kernel $T^0(E)$ of the reduction map

$$\mathrm{T}_pE \to \mathrm{T}_p\tilde{E}$$

where $\tilde{E}$ is the elliptic curve over $\mathbb{F}_p$ which is the reduction of E at p. This subrepresentation gives a local condition which can be used to define a Selmer complex as described below.

As above, let E be an elliptic curve defined over $\mathbb{Q}$ and let p be a prime number. Let Σ be a finite set of places of $\mathbb{Q}$ containing the infinite place, p and all places of bad reduction of E. Let K be a finite extension of $\mathbb{Q}$. Write Σ_K for the places of K above Σ. We often write Σ for Σ_K if there is no risk of confusion. Write $\mathrm{Gal}_{K,\Sigma}$ for the Galois group of the maximal extension of K which is unramified outside of Σ.

The Selmer complex $SC(T_pE, T^0(E), K)$ of E over K, with local condition $T^0(E)$, is defined as

$$\mathrm{Cone}\left(\begin{array}{c} C^\bullet(\mathrm{Gal}_{K,\Sigma}, T) \oplus \bigoplus\limits_{\nu|p} C^\bullet(\mathrm{Gal}_{K_\nu}, T^0(E)) \oplus \bigoplus\limits_{\nu\in\Sigma_f\setminus\{\nu|p\}} C^\bullet_f(\mathrm{Gal}_{K_\nu}, T) \\ \downarrow \\ \bigoplus\limits_{\nu\in\Sigma_f} C^\bullet(\mathrm{Gal}_{K_\nu}, T) \end{array}\right)[-1]$$

where Gal_{K_ν} is the absolute Galois group of K_ν, Σ_f are the finite places of Σ, $C^\bullet_f(\mathrm{Gal}_{K_\nu}, T)$ is the unramified cohomology (see definition 8.2.1) and the vertical maps are the restriction and the inclusions, respectively.

Let K_∞ be a Galois extension of $\mathbb{Q}$ such that

(i) $\mathrm{Gal}(K_\infty/\mathbb{Q})$ is a p-adic Lie group,

(ii) the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_{\mathrm{cyc}}$ of $\mathbb{Q}$ is contained in K_∞ and

(iii) K_∞ is unramified outside Σ.

Define

$$SC(T_pE, T^0(E), K_\infty) \coloneqq \varprojlim_K SC(T_pE, T^0(E), K)$$

where K runs through the finite extensions of $\mathbb{Q}$ which fulfil $\mathbb{Q} \subset K \subset K_\infty$.

Then it is known that the cohomology of $SC(T_pE, T^0(E), K_\infty)$ is finitely generated and it is conjectured to be torsion[1] over the Iwasawa algebra

$$\Lambda(\mathrm{Gal}(K_\infty/\mathbb{Q})) \coloneqq \varprojlim_K \mathbb{Z}_p[\mathrm{Gal}(K/\mathbb{Q})].$$

It is further conjectured by [CFKSV] and [FK06] that the cohomology of the Selmer complex $SC(T_pE, T^0(E), K_\infty)$ is S-torsion, for the canonical Ore set $S \subset \Lambda(\mathrm{Gal}(K_\infty/\mathbb{Q}))$ defined in [CFKSV]. This allows one to define a characteristic element for $SC(T_pE, T^0(E), K_\infty)$ and formulate a main conjecture as in [CFKSV].

Now let us turn to the situations when E has good supersingular reduction at p. Then the problematic twist is the fact that

$$\mathrm{V}_pE = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_pE$$

is an absolutely irreducible representation of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. Therefore we do not have a "local condition" $T^0(E)$ to define a Selmer complex with $\Lambda(\mathrm{Gal}(K_\infty/\mathbb{Q}))$-torsion cohomology groups. In addition, the p-adic L-function of E constructed by Amice-Vélu [AV75] and Višik [Viš76] when $K_\infty = \mathbb{Q}_{\mathrm{cyc}}$ is only a distribution on $\mathrm{Gal}(\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}_p)$ and not a measure. From now on we stick to $K_\infty = \mathbb{Q}_{\mathrm{cyc}}$. Nevertheless, in the case $K = \mathbb{Q}_{\mathrm{cyc}}$ Perrin-Riou [PR00] was able to construct a natural p-adic L-function in this setting which does not, however, arise as the characteristic element of any Selmer complex. For this reason, the

[1] The statement was initially conjectured by Mazur [Maz72] for the cyclotomic $\mathbb{Z}_p$-extension, Mazur's conjecture was proven by Rohrlich and Kato. The statement for pro-p extensions follows from Kato-Rohrlich [CS12].

Iwasawa theory of elliptic curves at supersingular primes looked very different from the much better understood Iwasawa theory at ordinary primes.

Recently, however, Pottharst [Pot13; Pot12] was able to construct a natural Selmer complex for primes of supersingular reduction in a manner similar to that for ordinary primes by replacing the role of Iwasawa algebras by the distribution algebras of Schneider and Teitelbaum [ST03].

We now briefly describe Pottharst's approach since it plays a key role for us. The idea is to use the fully faithful functor

$$\begin{pmatrix} \mathbb{Q}_p\text{-Galois representations} \\ \text{with } \mathbb{Q}_p\text{-coefficients} \end{pmatrix} \xrightarrow{\mathbf{D}^\dagger_{\mathrm{rig},\mathbb{Q}_p}} \begin{pmatrix} (\varphi,\Gamma)\text{-modules} \\ \text{over } \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p} \end{pmatrix}$$

where irreducible objects on the left hand side often become reducible on the right hand side. For an elliptic curve E, the $\mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}$-module $\mathbf{D}^\dagger_{\mathrm{rig},\mathbb{Q}_p}(\mathrm{V}_pE)$ is free of rank 2. The rank one (φ,Γ)-submodules of $\mathbf{D}^\dagger_{\mathrm{rig},\mathbb{Q}_p}(\mathrm{V}_pE)$ correspond to φ-stable rank one subspaces of $\mathbf{D}_{\mathrm{crys}}(\mathrm{V}_pE)$ (see [Pot13, §3.1]). The structure of $\mathbf{D}_{\mathrm{crys}}(\mathrm{V}_pE)$ is explicitly known; it is a two-dimensional $\mathbb{Q}_p$-vector space and we can choose a basis such that φ acts via the matrix

$$\begin{pmatrix} 0 & -1 \\ p & a_p \end{pmatrix}.$$

Then rank one subspaces correspond to φ-eigenspaces which correspond to the roots of the Frobenius polynomial

$$X^2 - a_pX + p.$$

Therefore for each root α of the Frobenius polynomial we have a rank one submodule D_α of $\mathbf{D}^\dagger_{\mathrm{rig},\mathbb{Q}_p}(\mathbb{Q}_p(\alpha)\otimes \mathrm{V}_pE)$. The subspace D_α fulfils the Dabrowski-Panchishkin property, i.e.

$$\mathbf{D}_{\mathrm{dR}}(D_\alpha) \cong t_{dR}(\mathbb{Q}_p(\alpha)\otimes \mathrm{V}_pE) \coloneqq \mathbf{D}_{\mathrm{dR}}(\mathbb{Q}_p(\alpha)\otimes \mathrm{V}_pE)/\mathbf{D}^0_{\mathrm{dR}}(\mathbb{Q}_p(\alpha)\otimes \mathrm{V}_pE).$$

This allows us to define the Selmer complex as before but now as a complex over Schneider-Teitelbaum's distribution algebra $D(\mathrm{Gal}(K_\infty/\mathbb{Q}),\mathbb{Q}_p(\alpha))$ rather than the Iwasawa algebra $\Lambda(\mathrm{Gal}(K_\infty/\mathbb{Q}))$. We note that $\Lambda(\mathrm{Gal}(K_\infty/\mathbb{Q}))[p^{-1}]$ is dense in $D(\mathrm{Gal}(K_\infty/\mathbb{Q}),\mathbb{Q}_p)$. Moreover, if E has good ordinary reduction at p and α is the unit root, then D_α is just $\mathbf{D}^\dagger_{\mathrm{rig},\mathbb{Q}_p}(\mathbb{Q}_p(\alpha)\otimes T^0(E))$.

We remark that the distribution algebra $D(G,K)$, which was defined by Schneider-

Teitelbaum [ST03, §4], for a p-adic Lie group G and a discretely valued field K can be alternatively defined using Berthelot's construction for A_∞ [BO78] (see proposition 3.5.4 for the canonical isomorphism between A_∞ and $D(G,K)$) as follows: let I be the Jacobson radical of

$$\Lambda = \Lambda_{\mathcal{O}_K}(G) \coloneqq O_K \otimes_{\mathbb{Z}_p} \Lambda(G).$$

Let π_K be a uniformiser of K. Then for $n \geq 1$ we define

$$\begin{aligned} &\Lambda_n^0 \coloneqq \Lambda[I^n/\pi_K] \subset \Lambda[\pi_K^{-1}], \\ &\Lambda_n \coloneqq I\text{-adic completion of } \Lambda_n^0 \qquad \text{and} \\ &A_n \coloneqq \Lambda_n[\pi_K^{-1}]. \end{aligned}$$

Lastly, to remove the dependence on n, we set

$$A_\infty \coloneqq \varprojlim A_n.$$

A key result of Pottharst is that Perrin Riou's p-adic L-function is a characteristic element of the Selmer complex defined with respect to the above algebra (see [Pot12]).

In a somewhat different direction, Fukaya and Kato generalised the classical formulation of main conjectures for elliptic curves with good ordinary reduction at p by formulating (in [FK06]) a natural main conjecture for any motive with good ordinary reduction at p and any compact p-adic Lie extension K_∞/K as above.

The main aim of the present thesis is now to firstly extend the approach of Pottharst to a non-commutative setting and then to combine this extended formalism with the seminal work of Fukaya and Kato in [FK06].

In this way we are able to formulate a natural main conjecture of Iwasawa theory for any motive that satisfies the Dabrowski-Panchishkin condition at p on the level of (φ, Γ)-modules (as discussed above) and any compact p-adic Lie extension of number fields.

In particular, by simultaneously extending the results of Pottharst and of Fukaya and Kato we shall thereby obtain a uniform way of formulating a natural GL_2 main conjecture (in the spirit of Coates et al. [CFKSV]) for elliptic curves that have either ordinary or supersingular reduction at p (see section 8.4).

Moreover, we are also able to show (in theorem 7.4.4) that the validity of our main conjecture follows from that of the appropriate cases of the Equivariant Tamagawa Number Conjecture (as discussed by Fukaya and Kato in [FK06, §2]) and of the Local Epsilon

Conjecture (as discussed in chapter 6).

Relationship with other Work Iwasawa theory for elliptic curves and primes of good supersingular reduction began with the work of Pollack [Pol03] and Kobayashi [Kob03] (assuming $a_p = 0$). This has been generalised to modular forms of higher weights and a_p not necessarily zero (see [Spr12],[LLZ10], also see [LZ12] for a non-commutative main conjecture for elliptic curves over $\mathbb{Q}$ and primes of good supersingular reduction with $a_p = 0$). All these works have focused on defining p-adic L-functions and torsion Selmer groups over the Iwasawa algebra. Thus the p-adic L-functions and Selmer groups have simpler analytic properties but their relationship with motivic invariants is somewhat intricate. Whereas the work of Pottharst, and therefore our generalisation of it, works with Selmer groups and p-adic L-functions that are directly related to motivic invariants and defined in a manner analogous to the ordinary situation. The price we have to pay is that we are forced to work with distribution algebras which are much more complicated than Iwasawa algebras. It is an interesting question to find relations between our work and that of [Pol03], [Kob03], [Spr12], [LLZ10] and [LZ12].

In the last chapter of this thesis we give a method, only in the commutative case, of recovering main conjectures over Iwasawa algebras from our main conjectures over distribution algebras. Thus we provide an alternate method of arriving at the conjectures in the above mentioned works. We hope our method will be amenable to non-commutative generalisations.

0.2 Content

We now discuss the contents of the individual chapters.

In chapter 1 we introduce some basic notations, concepts and constructions belonging to the area of non-commutative algebra, e.g. matrices, tensor products, determinant functors, non-commutative power series and polynomial rings. Furthermore we prove a specialised version of Nakayama's lemma and a non-commutative version of a standard flatness criterion. We also show some statements in homological algebra which play a crucial role in a perfectness result later on.

In chapter 2 we review the foundation of topologised spaces, such as K-Banach spaces, K-Fréchet spaces and K-Fréchet-Stein algebras. Furthermore we discuss the concept of orthonormalisable K-Banach spaces since they exhibit many favourable properties which we gainfully exploit in later chapters. A natural and very important finiteness condition for modules over a K-Fréchet-Stein algebra is that a module considered as a sheaf is locally finitely generated. Schneider-Teitelbaum call these modules coadmissable and we discuss properties of the full subcategory of coadmissable modules. Due to the weakening of the standard finitely generated hypothesis there are three natural notions of projective coadmissable objects. We relate them. Similarly there are two natural choices for the notion of a perfect complex, for the K-theory and for the determinant category respectively.

The first part of chapter 3 redevelops the theory of Tate algebras in a non-commutative setting, i.e. the non-commutative Tate algebra is the non-commutative power series where the coefficients are sequences which converge to zero. As in the commutative case we let nc-affinoids be quotients of the non-commutative Tate algebra. We try to recover some results of the classical theory but some fundamental results do not hold in the non-commutative case, e.g. the non-commutative Tate algebra is not noetherian. We review and extend Berthelot's construction of the generic fibre of formal spectra associated with, for example, Iwasawa algebras. Then we relate Berthelot's point of view with Schneider-Teitelbaum's (completed) distribution algebras and we deduce from Schneider-Teitelbaum's seminal work many very strong consequences. Based on these properties we are able to extend Pottharst's work on group cohomology over (completed) distribution algebras and we are able to establish the standard perfectness and base change results.

As a precursor for the chapter on (φ, Γ_K)-modules we generalise parts of Berger-Colmez's paper [BC08] in chapter 4 to non-commutative coefficients. This entails that we redevelop the classical Tate-Sen theory for (possibly non-commutative) orthonormalisable Banach algebras. Morally Tate-Sen theory states that a cocycle of G with values in a ring $\tilde{\Lambda}$ which fulfils the Tate-Sen axioms can be first base changed to a cocycle which is trivial on a large subgroup H of G. Then, by means of a further base change, one can restrict the values of the coycle to a certain dense subset ('decompletion') of $\tilde{\Lambda}^H$. We apply Berger-Colmez's Tate-Sen machinery to representations of G, i.e. by employing rings of p-adic Hodge theory as coefficients, one can simplify the Galois operation considerably since the Galois operation factors through the pro-cyclic group G/H. We then go on and prove standard compatibility results of this construction, like compatibility with base change, tensor products, etc.

In chapter 5 we redevelop the theory of (φ, Γ_K)-modules for (possibly non-commutative) orthonormalisable Banach algebras. Using chapter 4 we are indeed able to define a functor from local Galois representations to (φ, Γ_K)-modules. Furthermore we define the cohomology of a (φ, Γ_K)-module via the Herr complex and extend some standard results to our general setting. The Herr complex still computes the correct cohomology as we find that Pottharst's comparison of the cohomology of local Galois representations and the cohomology of the associated (φ, Γ_K)-module extends to the non-commutative case.

In order to be able to (conjecturally) define p-adic L-functions in the next chapter, the philosophy of Fukaya-Kato dictates that for the determinant of the cohomology of the local condition one needs a canonical trivialisation, which they call an ϵ-isomorphism. As indicated in the introduction we use $(\varphi, \Gamma_{\mathbb{Q}_p})$-modules as local conditions instead of local Galois representations. Hence, in chapter 6 we generalise Fukaya-Kato's non-commutative Local Epsilon Conjecture for local Galois representations and Nakamura's commutative Local Epsilon Conjecture for $(\varphi, \Gamma_{\mathbb{Q}_p})$-modules to a non-commutative Local Epsilon Conjecture for $(\varphi, \Gamma_{\mathbb{Q}_p})$-modules.

In chapter 7 we (conjecturally) define a p-adic L-function in the most general situation, i.e. we are given a global Galois representation with coefficients in an Iwasawa algebra and over the associated distribution algebra a local condition which is just a map whose codomain is the associated (φ, Γ_K)-module of the global Galois representation. Furthermore, one defines a Selmer complex in the usual way. Then, using the conjectural definition, one can deduce the main conjecture and also the exact values of the p-adic L-function at motivic points.

In chapter 8 we consider the p-adic L-function of a motive which fulfils the Dabrowski-Panchishkin condition on the level of (φ, Γ_K)-modules. We then deduce more explicit special value formulas at twisted Artin characters and we also compute the case of a modular form explicitly.

Using results of Lazard we are able to associate to the p-adic L-function an element in a quotient of the distribution algebra in chapter 9 if the Galois group is sufficiently simple. We also prove results about the uniqueness of the lift in the distribution algebra. In the case of a modular form we can recover Mazur-Tate-Teitelbaum's p-adic L-function and using the Bézout property we are also able to give a different point of view on the $\pm$-construction of Pollack.

0.3 Notation

We fix a prime number p. A ring in this thesis means a non-zero, associative, unital ring, potentially non-commutative. A ring is left (right) noetherian if all chains of left (right) ideals fulfil the ascending chain condition, i.e. become stationary. i.e. for left ideals A module is usually a left module if not stated otherwise. A left (right) module is left (right) noetherian if all chains of submodules fulfil the ascending chain condition, i.e. become stationary.

Unless stated otherwise, K is a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$ and residue field k.

For us complete means complete and Hausdorff. We topologise the set of homomorphisms between two topological spaces using the compact-open topology.

1 Non-Commutative Algebra

We review and in some cases extend a few basic algebraic results which are fundamental for the later chapters.

1.1 Rings and Algebras

Definition 1.1.1. Let R be a ring, then a ring A is an R-*ring* if A is a ring which is additionally an R-bimodule and the multiplication is left R-linear in the first argument and right R-linear in the second argument.

An R-ring A is called an R-*algebra* if R maps to the centre of A. Here we implicitly assume that R is commutative.

1.2 Matrices

Let R be a not necessarily commutative ring.

We want to quickly explore some properties of the matrix ring $M_d(R)$. The usual product on $M_d(R)$ given by $A \cdot B \coloneqq (\sum_j A_{i,j} \cdot B_{j,k})_{i,k}$ is associative, distributive and fulfils

$$\begin{aligned} &r \cdot (A \cdot B) = (r \cdot A) \cdot B, \\ &(A \cdot B) \cdot r = A \cdot (B \cdot r), \\ &(A \cdot r) \cdot B = A \cdot (r \cdot B) \\ &\text{and } I_d \cdot A = A \cdot I_d = A. \end{aligned}$$

We have the following modification of a well-known commutative statement:

Lemma 1.2.1 ([FD93, Prop. 1.7])**.** *Let V be a free (left) R-module of rank d with a chosen basis. Then the basis induces a ring isomorphism $\eta : \mathrm{End}_R(V) \to M_d(R^{\mathrm{op}})$ which restricts to a group isomorphism $\eta : \mathrm{Aut}_R(V) \to \mathrm{GL}_d(R^{\mathrm{op}})$.*

1.3 Tensor Products

Remark 1.3.1. Let R be a ring, M a right R-module and N a left R-module. Then we can define the tensor product $M \otimes_R N$ explicitly in the usual way. It is an abelian group and there is a canonical map $M \times N \to M \otimes_R N$. Moreover, there is the traditional universal mapping property and if M is an R_1-R_2-bi-module and N an R_2-R_3-bi-module, then $M \otimes_{R_2} N$ is an R_1-R_3-bi-module. Note that every left or right module is a $\mathbb{Z}$-$\mathbb{Z}$-bi-module. Furthermore, the tensor product fulfils a natural associativity relation.

Remark 1.3.2. If S and T are R-algebras, then $S \otimes_R T$ also has a structure of an R-algebra.

1.4 Chain Complexes

A complex is said to be $*$-bounded for $* \in \{[n,m], b, +, -, \varnothing\}$ if it is concentrated in $[n,m]$, bounded, bounded below, bounded above, or arbitrary. We denote the category of $*$-bounded R-module complexes by $\mathbf{K}^*(R)$. Let $\mathbf{D}(R)$ be the derived category of R-modules (see [Ver96]) and let $\mathbf{D}^*(R)$ denote the full subcategory of objects which can be represented by complexes in $\mathbf{K}^*(R)$.

We define $\mathbf{D}^*_{\mathrm{ft}}(R)$ to be the full subcategory of $\mathbf{D}^*(R)$ consisting of objects whose cohomology modules are finitely generated R-modules. Furthermore, we define $\mathbf{D}^{[a,b]}_{\mathrm{perf}}(R)$ to be the full subcategory of $\mathbf{D}^{[a,b]}(R)$ consisting of objects which are quasi-isomorphic to a complex of projective, finitely generated modules concentrated in degrees $[a,b]$ and we let $\mathbf{D}_{\mathrm{perf}}(R)$ be the full subcategory of $\mathbf{D}(R)$ consisting of objects which are quasi-isomorphic to a bounded complex of projective, finitely generated R-modules.

We say that a complex has a (P) resolution if the complex is isomorphic in the derived category to a bounded above complex whose entries all have property (P).

1.5 K-Theory and Determinant Categories

For an exact category $\mathcal{E}$ (in the sense of [Qui75, p. 91]) we can associate the K-theory groups $K_i(\mathcal{E})$ (for $i \geq 0$) and the *category of virtual objects* $\mathbf{Det}(\mathcal{E})$ (see [Qui75] and [BF03, §2]). Note that there are isomorphisms

$$K_i(\mathcal{E}) \xrightarrow{\sim} \pi_i(\mathbf{Det}(\mathcal{E}))$$

for $i = 0, 1$. One important example for an exact category is the category $\mathrm{PMod}(R)$ of finitely generated, projective modules over a (possibly non-commutative) ring R. See

[FK06, §1.2] for a simplified construction of the K-groups and the determinant category in this special case. We will systematically drop PMod from the notation, e.g. we will write $K_i(R)$ instead of $K_i(\mathrm{PMod}(R))$.

There is a canonical determinant functor

$$\mathbf{d}_{\mathcal{E}} : \mathcal{E} \longrightarrow \mathbf{Det}(\mathcal{E}),$$

(see [BF03, §2.3]). For an automorphism f of T, where T is an object of $\mathcal{E}$, we sometimes write $[\mathbf{d}_{\mathcal{E}}\, T, f]$, shortened to $[T, f]$, for $\mathbf{d}_{\mathcal{E}}\, f$.

Note that in the case of a ring R the above determinant functor extends naturally to a functor from $\mathbf{D}_{\mathrm{perf}}(R)$ to $\mathbf{Det}(R)$ (see [BF03, Prop. 2.1]).

Over *commutative* rings one can also consider the classical construction of the graded line bundle determinant category $\mathbf{Det}^{\mathrm{KM}}(R)$ (see [KM76] and [BF03, §2.5] for more details). The associated graded line bundle to a finitely generated projective R-module will be denoted by $\det_R T$ where T is a finitely generated, projective R-module. There exists a unique projection functor

$$\det\nolimits_R^{\mathrm{KM}} : \mathbf{Det}(R) \to \mathbf{Det}^{\mathrm{KM}}(R).$$

The projection functor is an equivalence of categories if and only if the objects and morphisms encode the same information, i.e. if and only if the canonical maps

$$K_0(R) \longrightarrow H^0(\mathrm{Spec}(R), \mathbb{Z}) \times \mathrm{Pic}(R) \qquad \text{and} \tag{1.5.1}$$

$$K_1(R) \longrightarrow R^\times \tag{1.5.2}$$

are isomorphisms (see [BF03, p. 509]). Known classes of rings fulfilling these properties are local rings or the ring of integers of a number field.

1.6 Polynomials and Power Series

Definition 1.6.1. Let $R\langle\langle X_1, \dots, X_n\rangle\rangle$ be the *non-commutative power series ring over* R with *non-commuting* indeterminates $X_1, \dots, X_n$ (which however commute with R), i.e. a general element looks like $f(X) = \sum_I c_I X^I$ with $c_I \in R$ and I runs over all elements of the free monoid[1] on the set $\{1, \dots, n\}$ which we denote by $\{1, \dots, n\}^*$.

It has the subring $R\langle X_1, \dots, X_n\rangle$ of the *non-commutative polynomial ring over* R where almost all coefficients vanish.

[1] The free monoid of a set consists of all finite strings which can be formed using the given elements.

Assuming there is a topology on R, the *non-commutative restricted power series ring* $R\{|X_1, \ldots, X_n|\}$ *over* R is the subring consisting of power series where the coefficients c_I converge to 0 for $\#I \to \infty$, where $\#I$ is the length of the string I.

Note that these rings are not R-algebras, unless R is commutative.

Definition 1.6.2. For a commutative ring R, an R-algebra A is called *nc-finite type*, short *ncft*, if for some $n \in \mathbb{N}$ there is a surjection $R\langle X_1, \ldots, X_n\rangle \twoheadrightarrow A$, extending the structure morphism.

1.7 An I-adic Version of Nakayama's Lemma

The following is a version of Nakayama's lemma and we use the standard proof.

Lemma 1.7.1. *Let R be a commutative ring and $I \subset R$ be a nilpotent ideal, i.e. there is an m such that $I^m = 0$. Let $\varphi : N \to M$ be an R-module map such that the induced map $\overline{\varphi} : N \to M/I$ is surjective. Then $\varphi : N \to M$ is also surjective.*

Proof. Define $\tilde{N}$ to be the image of φ in M. From the assumption immediately follows

$$M = \tilde{N} + IM$$

and iterating yields

$$M = \tilde{N} + IM = \tilde{N} + I\left(\tilde{N} + IM\right) = \tilde{N} + I\tilde{N} + I^2M = \tilde{N} + I^2M = \ldots = \tilde{N} + I^nM$$

as $\tilde{N}$ is an R-submodule of M. As I is assumed to be nilpotent, $\operatorname{im}\varphi = \tilde{N} = M$. □

We now prove a well known I-adic version of Nakayama's lemma which we were, however, unable to locate in the literature.

Corollary 1.7.2. *Let R be a commutative ring which is I-adically complete, in particular separated. Let $\varphi : N \to M$ be an R-module map of I-adically complete R-modules and assume that the induced map $\overline{\varphi} : N \to M/I$ is surjective. Then $\varphi : N \to M$ is also surjective.*

Proof. The map $\varphi/I^n : N/I^n \to M/I^n$ is surjective by the above lemma using $R' = R/I^n$, $N' = N/I^n$, $M' = M/I^n$ and $I' = \overline{I}$ which is visibly nilpotent in $R' = R/I^n$.

Hence the limit map

$$\varphi : N \cong \varprojlim N/I^n \to \varprojlim M/I^n \cong M$$

is surjective provided we can show that the system $\ker(N/I^n \to M/I^n)_n$ has the Mittag-Leffler property. As the functor $R/I^n \otimes_R -$ is right exact, we find that $\ker(N \to M)/I^n$ surjects on $\ker(N/I^n \to M/I^n)$. The system $(\ker(N \to M)/I^n)_n$ has surjective transition maps, hence $\ker(N/I^n \to M/I^n)_n$ also has surjective transition maps and thus the projective system clearly has the Mittag-Leffler property. $\square$

1.8 Local Criterion of Flatness

Definition 1.8.1. Let R be a ring with a two-sided ideal I. We say that I has the left *Artin-Rees property* if for every left ideal $\mathfrak{a}$ there is a non-negative k such that $\mathfrak{a} \cap I^k \subseteq I\mathfrak{a}$. The right Artin-Rees property is defined symmetrically.

Definition 1.8.2. Let R be a ring with a two-sided ideal I. We say that a (left) R-module M is *idealwise separated for* I if for every right ideal $\mathfrak{a}$ the intersection of $[\mathfrak{a}I^n \otimes_R M]$ for $n \geq 1$ in $\mathfrak{a} \otimes_R M$ is zero where $[-]$ denotes the image in $\mathfrak{a} \otimes_R M$.

Remark 1.8.3. If R is a commutative ring we are interested in $I^n(\mathfrak{a} \otimes_R M) \subset \mathfrak{a} \otimes_R M$, however if R is non-commutative $\mathfrak{a} \otimes_R M$ just has the structure of an abelian group, so we cannot form the subset. Our replacement is the image of $\mathfrak{a}I^n \otimes_R M$ in $\mathfrak{a} \otimes_R M$, which can serve the same role (see the next lemma). We note that the situation is symmetric, i.e. we have $[\mathfrak{a}I^n \otimes_R M] = [\mathfrak{a} \otimes_R I^n M]$ in $\mathfrak{a} \otimes_R M$, even though $\mathfrak{a}I^n \otimes_R M$ and $\mathfrak{a} \otimes_R I^n M$ are not isomorphic in general.

The next proposition is the non-commutative version of [Mat80, Thm. 20.C]. Beforehand we note that also in the non-commutative case it suffices to test left (right) flatness of a module just on right (left) ideals, which might be even chosen to be finitely generated, see [Lam99, Modified Flatness Test (4.12)].

Proposition 1.8.4. *Let R be a ring with a two-sided ideal I and let M be a (left) R-module which is idealwise separated for I. Assume furthermore that I^n has the right Artin-Rees property for all n and M/I^n is flat over R/I^n for all n, then M is flat over R.*

Proof. As mentioned above it suffices to show that for every right ideal $\mathfrak{a}$ the map $j : \mathfrak{a} \otimes_R M \to M$ is injective. And since M is I-adically separated, it suffices to show that $\ker j \subseteq [\mathfrak{a}I^n \otimes_R M]$ holds for all n.

As I^n has the right Artin-Rees property, we find a k such that $\mathfrak{a} \cap I^k \subseteq \mathfrak{a}I^n$. The map

$$\big(\mathfrak{a}/(\mathfrak{a} \cap I^k)\big) \otimes_R M \cong \big(\mathfrak{a}/(\mathfrak{a} \cap I^k)\big) \otimes_{R/I^k} M/I^k \to M/I^k$$

is injective because M/I^k is flat over R/I^k and the map $\mathfrak{a}/(\mathfrak{a} \cap I^k) \to R/I^k$ is injective. Then we consider the inclusions

$$\begin{aligned}
\ker j &\subseteq \ker\Big(\mathfrak{a} \otimes_R M \to M \to M/I^k\Big) \\
&= \ker\Big(\mathfrak{a} \otimes_R M \to \Big(\mathfrak{a}/(\mathfrak{a} \cap I^k)\Big) \otimes_R M \hookrightarrow M/I^k\Big) \\
&= \ker\Big(\mathfrak{a} \otimes_R M \to \Big(\mathfrak{a}/(\mathfrak{a} \cap I^k)\Big) \otimes_R M\Big) \\
&\subseteq \ker\Big(\mathfrak{a} \otimes_R M \to (\mathfrak{a}/\mathfrak{a}I^n) \otimes_R M = (\mathfrak{a} \otimes_R M) \,/\, [\mathfrak{a}I^n \otimes_R M]\Big) \\
&= [\mathfrak{a}I^n \otimes_R M]
\end{aligned}$$

which show the desired statement. □

Remark 1.8.5. In the commutative situation the proof of [BO78, Lem. B.2.2] gives a slightly different point of view.

1.9 Homological Algebra

Here we record some statements in homological algebra which we are going to need below.

However, we first recall the following definitions:

Definition 1.9.1. The *mapping cone* $\mathrm{Cone}(f)$ of a morphism $f : A^\bullet \to B^\bullet$ of complexes is defined to be

$$\mathrm{Cone}(f) := A^\bullet[1] \oplus B^\bullet,$$

i.e. $\mathrm{Cone}(f)_n = A^{n+1} \oplus B$, together with the standard differential. Then the *mapping fibre* $\mathrm{Fib}(f)$ *of* f is the shifted cone $\mathrm{Cone}(f)[-1]$.

A Statement by Bethelot-Ogus We note that [BO78, Lem. B.8] is independent of the main text and even works in the non-commutative, non-flat, noetherian setting. For the convenience of the reader we state the modified version:

Proposition 1.9.2. *Let R be a (possibly non-commutative) ring and I a two-sided ideal. We assume R to be I-adically complete and $\mathrm{gr}_I R$ to be noetherian. Let $C^\bullet$ be a complex left R-modules such that*

a) $H^*(\mathrm{gr}_I C^\bullet)$ is finitely generated as $\mathrm{gr}_I R$-module, and

b) the natural map $C^\bullet \to \varprojlim C^\bullet/I^n$ is a quasi-isomorphism.

Then the following are true:

(i) the inverse system $H^*(C^\bullet/I^n)$ has the Mittag-Leffler property,

(ii) the natural maps $H^*(C^\bullet) \to \varprojlim H^*(C^\bullet/I^n)$ are isomorphisms, and

(iii) all $H^*(C^\bullet)$ are finitely generated left R-modules.

The proof is a verbatim copy of [BO78, Lem. B.8].

Flat Canonical Truncation We are often in a situation where we would like to know if the canonical truncation of a complex of flat modules consists of flat modules again.

Lemma 1.9.3. *Let $C^\bullet$ be a complex of flat left R-modules such that for some a the group $H^{a-1}(M \otimes_R C^\bullet)$ vanishes for all finitely generated right modules M. Then the canonical truncation*

$$\tau_{\geq a}C^\bullet \qquad \cdots \longrightarrow 0 \longrightarrow \operatorname{coker}(C^{a-1} \to C^a) \longrightarrow C^{a+1} \longrightarrow \cdots$$

consists of flat R-modules. If additionally $H^a(C^\bullet)$ and $H^{a+1}(C^\bullet)$ vanish, then the canonical truncation

$$\tau_{\leq a+1}C^\bullet \qquad \cdots \longrightarrow C^a \longrightarrow \ker(C^{a+1} \to C^{a+2}) \longrightarrow 0 \longrightarrow \cdots$$

consists of flat R-modules.

Proof. Regarding the first claim, we note that

$$C^{a-2} \longrightarrow C^{a-1} \longrightarrow C^a \longrightarrow \operatorname{coker}(C^{a-1} \to C^a)$$

can be extended to a flat resolution of $\operatorname{coker}(C^a \to C^{a+1})$ because the above sequence is exact at C^{a-1} by assumption and exact at C^a by construction. Hence, for a finitely generated right R-module M we find

$$\begin{aligned}\operatorname{Tor}_1^R(M, \operatorname{coker}(C^{a-1} \to C^a)) &= H(M \otimes_R C^{a-2} \to M \otimes_R C^{a-1} \to M \otimes_R C^a)\\ &= H^{a-1}(M \otimes C^\bullet) = 0,\end{aligned}$$

i.e. $\operatorname{coker}(C^{a-1} \to C^a)$ is flat.

Regarding the second claim, we just note that $H^a(C^\bullet) = 0$ and $H^{a+1}(C^\bullet) = 0$ imply

$$\begin{aligned}\ker(C^{a+1} \to C^{a+2}) &= \operatorname{im}(C^a \to C^{a+1}) \\ &\cong C^a/\ker(C^a \to C^{a+1}) = C^a/\operatorname{im}(C^{a-1} \to C^a) = \operatorname{coker}(C^{a-1} \to C^a).\end{aligned}$$

□

Sometimes we have a complex of flat modules which is not bounded above. A priori we cannot use such a complex to compute the derived tensor product because it is not a flat resolution. However, the following corollary identifies a situation where we can indeed compute the derived tensor product using a potentially unbounded complex.

Corollary 1.9.4. *Let $C^\bullet$ be a complex of flat left R-modules such that $H^i(M \otimes_R C^\bullet)$ vanishes for $i \geq a$ and all finitely generated right modules M. Then $\tau_{\leq i} C^\bullet$ for any $i \geq a+2$ is a flat resolution of $C^\bullet$ and*

$$M \otimes^L_R [C^\bullet] = [M \otimes_R C^\bullet]$$

is an equality in the derived category $\mathbf{D}(R)$.

Proof. Fix an $i \geq a+2$. By the above lemma we have that $\tau_{\leq i} C^\bullet$ consists of flat modules. Moreover, the cohomology groups $H^j(C^\bullet)$ vanish for $j \geq a$, hence the canonical map $q_i : \tau_{\leq i} C^\bullet \to C^\bullet$ is a quasi-isomorphism. Thus $\tau_{\leq i} C^\bullet$ is indeed a flat resolution of $C^\bullet$.

Hence, the equality in the derived category follows since the homomorphism $M \otimes_R q_i$ is a quasi-isomorphism. Because the direct limit is exact and commutes with tensor products, it suffices to test the property on finitely generated modules M.

$M \otimes q_i$ induces indeed an isomorphism on $H^k(-)$ in degrees $k < i-1$ because these groups are left unchanged by the canonical truncation. The induced map for $k > i$ is also an isomorphism because on both sides the cohomology groups vanish. In degrees $k = i-1, i$ we have that $H^k(M \otimes_R C^\bullet)$ vanishes because $k > a$. Moreover $H^k(M \otimes_R \tau_{\leq i} C^\bullet)$ also vanishes because the functor $M \otimes_R -$ is right exact and $H^j(C^\bullet)$ vanishes for $j = i-1, i$. □

Perfect Complexes We recall that a complex is perfect if and only if it is a quasi-isomorphic to a bounded complex of finitely generated, projective R-modules. Our goal is to show the following non-commutative extension of a well-known result which we however are unable to locate in the literature.

Proposition 1.9.5. *Over a left noetherian ring R we find that a complex $C^\bullet$ of left R-modules is perfect if and only if the following conditions hold:*

(i) $H^i(C^\bullet)$ *is finitely generated for all* i*, and*

(ii) *the complex* $C^\bullet$ *has finite tor amplitude, i.e. for almost all* i *and all finitely generated right* R*-modules* N *the group* $H^i(N \otimes_R^L C^\bullet)$ *vanishes.*

More precisely we can say: assuming $C^\bullet$ *has tor amplitude in* $[a, b]$ *one can choose the complex of finitely generated, projective modules* $P^\bullet$ *to be concentrated in degrees* $[a, b]$*.*

We first need a lemma:

Lemma 1.9.6. *Let* R *be a left noetherian ring and* $C^\bullet$ *a complex of left* R*-modules such that*

(i) $H^i(C^\bullet)$ *is finitely generated for all* i*, and*

(ii) *the complex* $H^i(C^\bullet)$ *vanishes for* $i > b$*.*

Then $C^\bullet$ *is pseudo-coherent, i.e. it is quasi-isomorphic to a complex* $F^\bullet$ *consisting of finitely generated, free modules such that* $F^i = 0$ *for* $i > b$*.*

Proof. We can always find a surjection of a finitely generated free left module on a finitely generated left module, hence we can use the standard construction to generate finitely generated free resolutions of finitely generated left modules as all the occurring kernels are left submodules of finitely generated left modules, i.e. finitely generated because R is left noetherian.

Hence, let $F_i^\bullet$ be a finitely generated free resolution (concentrated in degrees $(-\infty, 0]$) of $H^i(C^\bullet)[0]$ (considered as a complex concentrated in degree 0) together with a quasi-isomorphism $\overline{q}_i^\bullet : F_i^\bullet \to H^i(C^\bullet)[0]$. As F_i^0 is free we can pick a lift $q_i^\bullet : F_i^0 \to \ker(C^i \to C^{i+1})[0] \subseteq C^i[0]$. Consider

$$\bigoplus_{i \le b} q_i^\bullet[-i] : F^\bullet := \bigoplus_{i \le b} F_i^\bullet[-i] \to C^\bullet$$

which is a quasi-isomorphism by construction. Furthermore, note that

$$F^k = \bigoplus_{i \le b} F_i^{k-i} = \bigoplus_{k \le i \le b} F_i^{k-i}$$

is a finitely generated free module. Here we used that if $i < k$, i.e. $k - i > 0$, then $F_i^{k-i} = 0$ because $F_i^\bullet$ is concentrated in degrees $(-\infty, 0]$. Thus $F^\bullet$ has all the desired properties. □

Proof of proposition 1.9.5. The "if" direction is obvious.

The complex $C^\bullet$ is quasi-isomorphic to a complex $F^\bullet$ such that all F_i are finitely generated, free left R-modules and $F^k = 0$ for $k > b$ by the previous lemma because $H^i(C^\bullet) = 0$ for $i \notin [a, b]$. The canonical truncation

$$P^\bullet = \tau_{\geq a} F^\bullet = \qquad \cdots \to 0 \to \operatorname{coker}(F^{a-1} \to F^a) \to F^{a+1} \to \cdots \to F^b \to 0 \to \cdots .$$

consists of flat modules by lemma 1.9.3 as $H^{a-1}(M \otimes_R F^\bullet) = H^{a-1}(M \otimes_R^L C^\bullet)$ vanishes. We note that $P^a = \operatorname{coker}(F^{a-1} \to F^a)$ is finitely presented. Because flatness and finite presentation imply projectivity (see [Wei94, Thm. 3.2.7]) we can conclude that $P^\bullet$ indeed has all required properties. □

2 Banach and Fréchet Spaces

We review the notions of non-archimedean K-Banach and K-Fréchet spaces where K is a finite extension of $\mathbb{Q}_p$.

2.1 Non-Archimedean Banach/Fréchet Spaces

The following is standard material which is nicely treated in [Sch02], additional references are [BGR84] and [Bel15, App. A].

Definition 2.1.1 ([Sch02, §2])**.** Let M be a K-vector space. A *semi-norm* is a function $q : M \to \mathbb{R}_{\geq 0}$ such that:

(i) $q(a \cdot m) = |a| \cdot q(m)$ for any $a \in K$, and

(ii) $q(m_1 + m_2) \leq \max(q(m_1), q(m_2))$, the so called ultra-metric property.

We say that a semi-norm is a *norm* if the semi-norm additionally fulfils $q(m) = 0$ if and only if $m = 0$.

Remark 2.1.2 ([Sch02, §4])*.* Assume that M can be (linearly) topologised with respect to a family of semi-norms $(q_i)_{i \in I}$. A basis of the topology is given by

$$B_F(m, \varepsilon) \coloneqq \{m' \in M | \forall i \in F : q_i(m - m') < \varepsilon\} = \{m' \in M | q_F(m - m') < \varepsilon\}$$

where we consider all *finite* subsets F of I and where we define q_F to be $\max_{i \in F} q_i$.

There is also the following different characterisation of the above topology: it is the coarsest topology such that each $q_i : M \to \mathbb{R}$ is continuous and all translation maps $m \mapsto m + m_0$ are continuous.

Definition 2.1.3 ([Sch02, §4/§8])**.** If the topology of M is induced by semi-norms, then we call M a K*-locally convex space.* If additionally the index set of semi-norms is countable and M is complete (see [Sch02, §7]), we call M a K*-Fréchet space.*

If M is topologised by one (semi-)norm, then we call M a K*-(semi-)normed space.* If M is additionally complete (and Hausdorff) we call it a K*-Banach space.* In this case the single semi-norm $q(-)$ is actually a norm usually denoted by $|{-}|$ since M is Hausdorff.

Remark 2.1.4. From now on all (index) sets of semi-norms $(q_i)_I$ of K-Fréchet spaces are implicitly considered to be countable.

We would like to derive an ε-δ-criterion for continuity.

Remark 2.1.5. A K-linear map of K-locally convex spaces $f : M \to N$ is continuous by definition if and only if f^{-1} applied to an open set is open. We only have to check continuity at 0. Furthermore it suffices to check continuity on the basis of the topology. Let $(q_i)_I$ and $(q'_j)_J$ be the semi-norms of M and N respectively. Then f is continuous if and only if for every basic open set $B_F(0, \varepsilon)$ where F is a finite subset of J and $\varepsilon > 0$, there is a finite set $G = G(F, \varepsilon) \subseteq I$ and $\delta = \delta(F, \varepsilon) > 0$ such that $B_G(0, \delta) \subseteq f^{-1}(B_F(0, \varepsilon))$. More classically this can be expressed as

$$q_G(m) < \delta \Rightarrow q'_F(f(m)) < \varepsilon.$$

Definition 2.1.6. Assuming that M is a K-Banach space we define the *integral elements $\mathcal{O}_M$ of M* as all elements $m \in M$ which fulfil $|m| \leq 1$.

Definition 2.1.7 ([Sch02, §5B])**.** Let N be a subspace of a K-locally convex space M and q a semi-norm on M, then we define the *quotient semi-norm* $\overline{q}$ to be

$$\overline{q}(m + N) := \inf_{n \in N} q(m + n).$$

The quotient topology on M/N is defined by the semi-norms $(\overline{q_F})_{F \subset I, \text{finite}}$.

Proposition 2.1.8 ([Sch02, Prop. 8.3])**.** *Let N be a closed subspace of a K-Fréchet space M. Then the quotient space M/N also has the structure of a K-Fréchet space.*

We have the following variant of the classical open mapping theorem:

Theorem 2.1.9 ([Sch02, Prop. 8.6])**.** *A continuous surjection of K-Fréchet spaces is open.*

Furthermore, as K in our setting is discretely valued, the following holds:

Proposition 2.1.10 ([Sch02, Prop. 10.5])**.** *Let M be a K-Banach space and N a closed subspace. Then the quotient morphism $M \twoheadrightarrow M/N$ admits a continuous K-linear section.*

Definition 2.1.11 ([Sch02, §17B])**.** Let A be a K-algebra and let M and N be right and left A-modules respectively which are also K-locally convex spaces with semi-norms $q : M \to \mathbb{R}$ and $q' : N \to \mathbb{R}$ respectively.

Then we define the *tensor product semi-norm* $q \otimes_A q'$ *on* $M \otimes_A N$ to be

$$(q \otimes_A q')(x) := \inf \left\{ \max_i q(m_i) q'(n_i) \mid x = \sum m_i \otimes n_i \in M \otimes_A N \right\}$$

for $x \in M \otimes_A N$.

Definition 2.1.12. We can equip $M \otimes_A N$ with the topology induced by the above semi-norms and call it the *projective tensor product topology.* We denote the Hausdorff completion, cf. [Sch02, Prop. 7.5], of $M \otimes_A N$ by $M \widehat{\otimes}_A N$ and name it the *completed tensor product of* M *and* N.

Remark 2.1.13. Assuming that M and N are K-Fréchet spaces one can say even more about the projective tensor product topology of $M \otimes_K N$ (see [Sch02, §17]). Furthermore, the completed tensor product is obviously a K-Fréchet space as one can describe the topology using the extension of the semi-norms $q_{M,i} \otimes_A q_{N,j}$ (see [Sch02, Rem. 7.4i]).

Lemma 2.1.14. *Let* A *be a* K*-algebra and* M *and* N *be right and left* A*-modules respectively. Assume that both modules are topologised using countably many semi-norms* $q_{M,i}$ *and* $q_{N,j}$ *respectively, that* A *is a topological ring and that the multiplication with* a *is continuous for all* $a \in A$*. Then the natural map*

$$M \widehat{\otimes}_A N \to \hat{M} \widehat{\otimes}_A N$$

is an isomorphism.

Proof. There is the canonical continuous map $c_M : M \to \hat{M}$ of A-modules which induces the continuous map $\alpha : M \otimes_A N \to \hat{M} \otimes_A N$. Its completion induces the continuous map $\hat{\alpha} : M \widehat{\otimes}_A N \to \hat{M} \widehat{\otimes}_A N$.

The canonical map c_M has dense image, i.e. every $\hat{m} \in \hat{M}$ can be written as $\lim_k c_M(m_k)$. Note that the sequence $(m_k)_k$ is also a Cauchy sequence in M. We define the map

$$\begin{aligned} \tilde{\beta} : \quad \hat{M} \times N &\longrightarrow M \widehat{\otimes}_A N \\ (\hat{m}, n) &\longmapsto \lim_k c_{M \otimes_A N}(m_k \otimes n). \end{aligned}$$

The limit on the right hand side converges because the canonical map $c_{M \otimes_A N}$ is continuous and it does not depend on the choice of the sequence $(m_k)_k$, because zero sequences will get mapped to zero in the Hausdorff space $M \widehat{\otimes}_A N$. As $\tilde{\beta}$ fulfils $\tilde{\beta}(\hat{m} \cdot a, n) = \tilde{\beta}(\hat{m}, a \cdot n)$ the map factors through the map $\beta : \hat{M} \otimes_A N \to M \widehat{\otimes}_A N$ by the universal property of the tensor product. Looking at the definition of the tensor product of semi-norms we see that if we want to show the continuity of β it suffices to proof the inequality

$$(q_{M,i} \mathbin{\widehat{\otimes}_A} q_{N,j}) \circ \beta(\hat{m} \otimes n) \leq q_{\hat{M},i}(\hat{m}) \cdot q_{N,j}(n).$$

Indeed:

$$\begin{aligned}
(q_{M,i} \mathbin{\widehat{\otimes}_A} q_{N,j}) \circ \beta(\hat{m} \otimes n) &= (q_{M,i} \mathbin{\widehat{\otimes}_A} q_{N,j}) \circ \beta((\lim_{k\to\infty} c_M(m_k)) \otimes n) \\
&= (q_{M,i} \mathbin{\widehat{\otimes}_A} q_{N,j})(\lim_{k\to\infty} c_{M \otimes N}(m_k \otimes n)) && \text{definition of } \beta \\
&= \lim_{k\to\infty} (q_{M,i} \mathbin{\widehat{\otimes}_A} q_{N,j}) \circ c_{M \otimes N}(m_k \otimes n) && \text{semi-norm cts} \\
&= \lim_{k\to\infty} (q_{M,i} \otimes_A q_{N,j})(m_k \otimes n) && \text{extension property} \\
&\leq \lim_{k\to\infty} q_{M,i}(m_k) \cdot q_{N,j}(n) && \text{definition of tensor semi-norm} \\
&= \lim_{k\to\infty} q_{\hat{M},i} \circ c_M(m_k) \cdot q_{N,j}(n) && \text{extension property} \\
&= q_{\hat{M},i}(\lim_{k\to\infty} c_M(m_k)) \cdot q_{N,j}(n) && \text{semi-norm cts} \\
&= q_{\hat{M},i}(\hat{m}) \cdot q_{N,j}(n).
\end{aligned}$$

Hence there is a unique continuous map $\beta' : \hat{M} \mathbin{\widehat{\otimes}} N \to M \mathbin{\widehat{\otimes}_A} N$ again by the universal property of the completion.

Now we have to show that the constructed maps $\hat{\alpha}$ and β' are indeed isomorphisms. We consider the commutative diagram

$$\begin{tikzcd}
M \mathbin{\widehat{\otimes}_A} N \arrow[r, "\hat{\alpha}"] & \hat{M} \mathbin{\widehat{\otimes}_A} N \arrow[r, "\hat{\beta}"] \arrow[dr, "\beta'"] & \widehat{M \mathbin{\widehat{\otimes}_A} N} \arrow[r, "\hat{\hat{\alpha}}"] & \widehat{\hat{M} \mathbin{\widehat{\otimes}_A} N} \\
M \otimes_A N \arrow[u, "c_{M \otimes_A N}"] \arrow[r, "\alpha"] \arrow[dr, equal, bend right] & \hat{M} \otimes_A N \arrow[u, "c_{\hat{M} \otimes_A N}"'] \arrow[r, "\beta"] \arrow[dr, equal] & M \mathbin{\widehat{\otimes}_A} N \arrow[u, "\wr", "c_{M \widehat{\otimes}_A N}"'] \arrow[r, "\hat{\alpha}"] & \hat{M} \mathbin{\widehat{\otimes}_A} N \arrow[u, "\wr", "c_{\hat{M} \widehat{\otimes}_A N}"'] \\
 & & M \otimes_A N \arrow[u, "c_{M \otimes_A N}"'] \arrow[r, "\alpha"'] & \hat{M} \otimes_A N \arrow[u, "c_{\hat{M} \otimes_A N}"']
\end{tikzcd}$$

where the canonical maps $c_{M \widehat{\otimes}_A N}$ and $c_{\hat{M} \widehat{\otimes}_A N}$ are isomorphisms because $M \mathbin{\widehat{\otimes}_A} N$ and $\hat{M} \mathbin{\widehat{\otimes}_A} N$ are complete. Hence

$$\hat{\beta} \circ \hat{\alpha} = \widehat{\beta \circ \alpha} = \hat{c}_{M \otimes_A N} = c_{M \widehat{\otimes}_A N}$$

and

$$\hat{\hat{\alpha}} \circ \hat{\beta} = \widehat{\hat{\alpha} \circ \beta} = \hat{c}_{\hat{M} \otimes_A N} = c_{\hat{M} \widehat{\otimes}_A N}$$

are isomorphisms. We deduce that $\hat{\alpha}$ and β' also are isomorphisms. ☐

Corollary 2.1.15. *Let A and B be K-algebras and assume that M is a right A-module, N is a A-B-bimodule and O is a left B-module. Furthermore we assume that the modules are K-Fréchet spaces, that the algebras are topological rings and that the multiplication with elements of A and B is continuous. Then the natural maps*

$$(M \widehat{\otimes}_A N) \widehat{\otimes}_B O \longleftarrow M \widehat{\otimes_A N \otimes_B} O \longrightarrow M \widehat{\otimes}_A (N \widehat{\otimes}_B O)$$

are isomorphisms.

Proof. We first note that the induced topology of $M \otimes_A N \otimes_B O$ does not depend on the presentation, i.e. $(M \otimes_A N) \otimes_B O$ and $M \otimes_A (N \otimes_B O)$ have the same topology given by semi-norms which are the obvious modification of definition 2.1.11 for three semi-norms. Hence the corollary follows from the previous lemma. ☐

Remark 2.1.16. We note that the definitions in this section also make sense for $\mathcal{O}_K$-modules, i.e. we can speak of $\mathcal{O}_K$-Banach algebras, etc.

Definition 2.1.17. We say that an $\mathcal{O}_K$-Banach algebra $\mathcal{A}$ is *π-adically complete as a $\mathcal{O}_K$-Banach algebra* if it is π-adically complete and in addition the norm $|-|_{\mathcal{A}}$ is the standard π-adic norm induced by the π-adic filtration, normalised such that $|\pi|_{\mathcal{A}} = |\pi|_K$ holds.

Remark 2.1.18. We note that this implies that $\mathcal{O}_K$-Banach algebras which are π-adically complete as a Banach algebra cannot have non-zero elements which are divisible by arbitrarily high powers of π. One obvious non-example is the $\mathbb{Z}_p$-Banach-algebra $\mathbb{Q}_p$ in which every element is divisible by p. However the completed $\mathbb{Z}_p$-algebra $\widehat{\mathbb{Z}_p[X]}$ with the norm given by $|X| = |p|_p = p^{-1}$, i.e. $|\sum a_i X^i| := \sup\left(|a_i| p^{-i}\right)$, is also not π-adically complete as a Banach algebra since p does not divide X.

2.2 Non-Archimedean Banach/Fréchet Algebras

Definition 2.2.1. A *K-Fréchet algebra S* is a K-algebra where the underlying K-vector space is a K-Fréchet space with the additional condition that the algebra multiplication is continuous.

If the set of semi-norms just consists of one norm, S is also called a *K-Banach algebra* if $|1| = 1$.

Remark 2.2.2. Note that S does not have to be commutative, i.e. we explicitly include non-commutative rings.

Remark 2.2.3. Assuming that S is a K-Banach algebra, then the integral elements $\mathcal{O}_S$ of S form a subring.

Definition 2.2.4. A left S-module M is called a left *K-Banach S-module* if

(i) S is a K-Banach algebra with norm $|{-}|_S$,

(ii) M is a K-Banach space with norm $|{-}|_M$, and

(iii) $|sm|_M \leq |s|_S |m|_M$.

There are analogous definitions for right and two-sided S-Banach modules.

Lemma 2.2.5. *We assume that A is a commutative topological K-algebra and that S and S' are K-Fréchet algebras which are also A-algebras. Let M and N be right and left $S \mathbin{\widehat{\otimes}}_A S'$-modules respectively which are also K-Fréchet modules. We require that the action of $S \mathbin{\widehat{\otimes}}_A S'$ on M and N is continuous, i.e. the maps $(S \mathbin{\widehat{\otimes}}_A S')^{\mathrm{op}} \to \mathrm{End}_{K,\mathrm{cts}}(M)$ and $S \mathbin{\widehat{\otimes}}_A S' \to \mathrm{End}_{K,\mathrm{cts}}(N)$ are continuous. Then there is the isomorphism*

$$M \otimes_{S \otimes_A S'} N \xrightarrow{\sim} M \otimes_{S \mathbin{\widehat{\otimes}}_A S'} N.$$

Proof. One just needs to show that the natural map

$$\alpha : M \times N \to M \otimes_{S \otimes_A S'} N$$

is $S \mathbin{\widehat{\otimes}}_A S'$-balanced, i.e. for every $t \in S \mathbin{\widehat{\otimes}}_A S'$ we have $\alpha(m \cdot t, n) = \alpha(m, t \cdot n)$. As the canonical map

$$c \coloneqq c_{S \otimes_A S'} : S \otimes_A S' \to S \mathbin{\widehat{\otimes}}_A S'$$

has dense image we can approximate t by $c(t_i)$ where $(t_i)_i$ is a sequence in $S \otimes_A S'$.

Furthermore we have $\alpha(m \cdot c(t_i), n) = \alpha(m, c(t_i) \cdot n)$. The statement now follows from observing that α is continuous in the first and second argument and that the actions on M and N are continuous, i.e. $m \cdot t = \lim m \cdot c(t_i)$ and $t \cdot n = \lim c(t_i) \cdot n$ holds. $\square$

Corollary 2.2.6. *Let A be a commutative topological K-algebra and let S and S' be K-Fréchet algebras which are also A-algebras. Moreover let M be a right S'-module and let N be a left $S\,\widehat{\otimes}_A\,S'$-module. We assume that the actions on M and N are continuous, i.e. the ring homomorphisms $(S')^{\mathrm{op}} \to \mathrm{End}_{K,\mathrm{cts}}(M)$ and $S\,\widehat{\otimes}_A\,S' \to \mathrm{End}_{K,\mathrm{cts}}(N)$ are continuous. Moreover we assume that the induced right action of $S\,\widehat{\otimes}_A\,S'$ on $S\,\widehat{\otimes}_A\,M$ is continuous. Lastly, we require the A-multiplication on S and S' to be continuous. Then the canonical homomorphism*

$$M\,\widehat{\otimes}_{S'}\,N \longrightarrow (S\,\widehat{\otimes}_A\,M)\,\widehat{\otimes}_{S\,\widehat{\otimes}_A\,S'}\,N$$

is an isomorphism.

Proof. Using the above lemma and corollary 2.1.15 we find:

$$\begin{aligned}(S\,\widehat{\otimes}_A\,M)\,\widehat{\otimes}_{S\,\widehat{\otimes}_A\,S'}\,N &\xrightarrow{\sim} (S\,\widehat{\otimes}_A\,M)\,\widehat{\otimes}_{S\otimes_A S'}\,N\\ &\xrightarrow{\sim} \widehat{(S\otimes_A M)\otimes_{S\otimes_A S'}}\,N\\ &\xrightarrow{\sim} M\,\widehat{\otimes}_{S'}\,N.\end{aligned}$$

□

2.3 Orthonormalisable Non-Archimedean Banach Spaces

We follow [Sch02, §10] and [Bel15, §A.3].

Definition 2.3.1. For an arbitrary set I and a K-Banach S-module M we define the K-Banach S-module $c_I(M)$ to be all functions $f : I \to M$ such that for all $\varepsilon > 0$ the set $\{x \in I \mid |f(x)| > \epsilon\}$ is finite. Its norm is given by the sup-norm $\sup_{x\in I}|f(x)|$.

Definition 2.3.2. A K-Banach S-module M is called *orthonormalisable* if there is an S-linear *isometry* $\varphi : c_I(S) \xrightarrow{\sim} M$. We call the set of elements $e_i = \varphi((\delta_{ij})_j)$, where δ_{ij} is the Dirac delta function, a *Schauder basis*. If φ is just a topological isomorphism we say that M is *potentially orthonormalisable.*

Recall that we assume that K is discretely valued.

Proposition 2.3.3 ([Ser62, §1],[Sch02, Prop. 10.1,Rem. 10.2])**.** *Every K-Banach space M is topologically isomorphic to $c_I(K)$ for some I. If additionally $|M| \subseteq |K|$ holds, then M is even isometric to $c_I(K)$, i.e. orthonormalisable.*

Then we have the following strengthening of proposition 2.1.10:

Corollary 2.3.4. *If M is a K-Banach space with $|M| \subseteq |K|$ and N is a closed subspace, then then there exists a continuous K-linear section of $p : M \twoheadrightarrow M/N$ of norm 1.*

Proof. By the proposition we find M/N is orthonormalisable and let (e_i) be the Schauder basis of $c_I(K) \cong M/N$. Note that

$$1 = |e_i| = \inf |p^{-1}(e_i)|.$$

Hence for every i and for every $\varepsilon > 0$ we find $v_i \in M$ such that $p(v_i) = e_i$ and $1 \leq |v_i| \leq 1+\varepsilon$. However the valuation of M is discrete, so for ε small enough we find $|v_i| = 1$ for all i. Fix such a set of v_i. Now employing the universal mapping property of $c_I(K)$ yields the desired section (see [Sch02, p. 66]). □

Lemma 2.3.5 ([Bel15, Lem. A.3.5])**.** *Assume that M is an orthonormalisable K-Banach space, i.e. there is an isometry $c_I(K) \xrightarrow{\sim} M$ and N is a K-Fréchet space with semi-norms (q_j). Let $c_I(N)$ be the Fréchet space of sequences $(n_i)_i$ indexed by $i \in I$ with values $n_i \in N$ such that $q_j(n_i) \to 0$ for $i \to \infty$ and for all all semi-norms q_j. Then $c_I(N)$ can be equipped with the semi-norms $\sup_i q_j(n_i)$. Then the natural K-linear map*

$$M \mathbin{\widehat{\otimes}}_K N \to c_I(N)$$

is an isomorphism of K-Fréchet spaces.

Corollary 2.3.6 ([Bel15, Cor. A.3.6])**.** *Let M be an orthonormalisable K-Banach space. Then $- \widehat{\otimes}_K M$ is an exact functor for K-Banach spaces.*

2.4 Fréchet-Stein Algebras and Coadmissable Modules

We follow [ST03].

Definition 2.4.1. Let q be a semi-norm of A, then $A/\{a \in A | q(a) = 0\}$ is a semi-normed space via q. Its completion is a K-Banach space which we denote by A_q.

Definition 2.4.2. Let A_∞ be a K-Fréchet algebra. It is called a (left) *K-Fréchet-Stein algebra* if there is a sequence $q_1 \leq q_2 \leq \ldots$ of continuous algebra semi-norms which define the Fréchet topology such that

(i) A_n is left noetherian, and

(ii) A_n is flat as a right A_{n+1} module via the canonical map $A_{n+1} \to A_n$

where A_n is the K-Banach space $(A_\infty)_{q_n}$. There are the obvious analogous right and two-sided K-Fréchet-Stein algebras.

The most prominent example in the class Fréchet-Stein algebra are the rigid analytic functions on the open unit disk:

Example 2.4.3. Let A_∞ be the power series in $\mathbb{Q}_p$ which converge on the open unit disk. Hence a power series $f = \sum a_k T^k$ belongs to A_∞ if and only if for all $0 \leq r < 1$ where $r \in p^{\mathbb{Q}}$ we have $a_k r^k \to 0$. In particular one can define norms

$$q_r(f) := \max_k |a_k|_p r^k.$$

Furthermore the completion of A_∞ with respect to the norm q_r are just the rigid analytic functions on the closed disk of radius r.

Definition 2.4.4. A *coherent sheaf* on a K-Fréchet-Stein algebra A_∞ with semi-norms $(q_n)_n$ is a family $(M_n)_n$ of finitely generated (left) modules M_n over A_n together with A_n-isomorphisms

$$A_n \otimes_{A_{n+1}} M_{n+1} \xrightarrow{\sim} M_n.$$

Definition 2.4.5. We may also define the *global section functor* $\Gamma\left((M_n)_n \right)$ as $\varprojlim M_n$.

We would like to single out a class of "good" modules over a K-Fréchet-Stein algebra:

Definition 2.4.6. We call a module over a K-Fréchet-Stein algebra *coadmissable* if it is isomorphic to the global section of some coherent sheaf.

Now we are equipped with all the notions to state the first important theorem in Schneider-Teitelbaum:

Theorem 2.4.7 ([ST03, §3]). *Let $(M_n)_n$ be a coherent sheaf over a K-Fréchet-Stein algebra and define M_∞ as $\Gamma\left((M_n)_n \right)$, then*

(i) (Theorem A) the natural map $M_\infty \to M_n$ has dense image, and

(ii) (Theorem B) all higher derived functors $\mathbf{R}^i \varprojlim M_n$ $(i \geq 1)$ vanish.

The theorem has important consequences:

Corollary 2.4.8 ([ST03, Rem. 3.2, Cor. 3.1, 3.5, 3.3, 3.4(v)]). *Using the notation as above:*

(i) A_n *is flat as an* A_∞*-module,*

(ii) *the natural morphism*

$$A_n \otimes_{A_\infty} M_\infty \longrightarrow M_n$$

is an isomorphism,

(iii) *the category of coadmissable* A_∞*-modules is abelian,*

(iv) *the category of coadmissable* A_∞*-modules is equivalent to the category of coherent sheaves over* A_∞*, and*

(v) *a finitely presented* A_∞*-module is coadmissable.*

Remark 2.4.9. We state here that we equip a coadmissable module $M_\infty = \varprojlim M_n$ with the projective limit topology where we assume that the finitely presented A_n-modules M_n are equipped with the standard K-Banach space topology. This topology makes M_∞ into a K-Fréchet space with continuous A_∞-multiplication.

2.5 Fréchet-Stein Algebras and the Notion of Projectivity

There are at least three notions of projective objects over K-Fréchet-Stein algebras since

(i) a module can be projective in the category of all modules,

(ii) the associated sheaf to a coadmissable module can consist of projective modules over every A_n, and

(iii) a module can be projective in the abelian category of coadmissable modules.

Obviously the first property implies the second one. We would like to find more relations between these notions.

Remark 2.5.1. In the next chapter (see Zábrádi's theorem 3.5.5) we will see that for Fréchet-Stein algebras associated with Iwasawa algebras of compact p-adic Lie groups actually all three notions of projectivity coincide.

Definition 2.5.2. We say a coadmissable module $M_\infty = \varprojlim M_n$ over a Fréchet-Stein algebra is *locally projective* if all M_n are projective over A_n.

Lemma 2.5.3. *For coadmissable modules $M_\infty = \varprojlim M_n$ and $N_\infty = \varprojlim N_n$ over a two-sided Fréchet-Stein algebra A_∞ we have the isomorphism*

$$\operatorname{Ext}^1_{A_\infty}(M_\infty, N_\infty) \xrightarrow{\sim} \varprojlim \operatorname{Ext}^1_{A_n}(M_n, N_n).$$

Proof. From [ST03, Lem. 8.3] follows the exact sequence

$$0 \longrightarrow \mathbf{R}^1 \varprojlim \operatorname{Hom}_{A_n}(M_n, N_n) \longrightarrow \operatorname{Ext}^1_{A_\infty}(M_\infty, N_\infty) \longrightarrow \varprojlim \operatorname{Ext}^1_{A_n}(M_n, N_n) \longrightarrow 0.$$

We would like to deduce the vanishing of the left hand side from theorem B (see theorem 2.4.7(ii)), hence we have to check that $\operatorname{Hom}_{A_n}(M_n, N_n)_n$ is a coherent sheaf. As M_n is finitely generated, we find a surjection $A_n^k \twoheadrightarrow M_n$. Thus $\operatorname{Hom}_{A_n}(M_n, N_n)$ is a submodule of $\operatorname{Hom}_{A_n}(A_n^k, N_n) = N_n^k$, which is finitely generated and we see that $\operatorname{Hom}_{A_n}(M_n, N_n)$ is finitely generated because A_n is noetherian. Furthermore the isomorphism

$$A_n \otimes_{A_{n+1}} \operatorname{Hom}_{A_{n+1}}(M_{n+1}, N_{n+1}) \xrightarrow{\sim} \operatorname{Hom}_{A_n}(M_n, N_n)$$

holds because A_n is flat over A_{n+1} and M_{n+1} is finitely presented. The proof of the previous statement is standard and can be found for example in [Eis95, Prop. 2.10]. □

Hence we can deduce that the second property implies the third property:

Corollary 2.5.4. *A locally projective module P_∞ over a two-sided Fréchet-Stein algebra is a projective object in the category of coadmissable modules.*

Proof. Let $M_\infty = \varprojlim M_n$ be a coadmissable module. Then all $\operatorname{Ext}^1_{A_n}(P_n, M_n)$ vanish because all P_n are projective, hence $\operatorname{Ext}^1_{A_\infty}(P_\infty, M_\infty)$ also vanishes. □

Sometimes we can go from the third to the first property:

Lemma 2.5.5. *Let A_∞ be a Fréchet-Stein algebra. If P_∞ is a* finitely generated *coadmissable A_∞-module which is a projective object in the category of coadmissable modules, then P_∞ is a projective A_∞-module.*

Proof. As P_∞ is finitely generated, there is a surjection of coadmissable modules $A_\infty^k \twoheadrightarrow P_\infty$. Because P_∞ is a projective object in the category of coadmissable modules the map $\operatorname{Hom}_{A_\infty}(P_\infty, A_\infty^k) \to \operatorname{Hom}_{A_\infty}(P_\infty, P_\infty)$ is also surjective. Hence P_∞ is a direct summand of A_∞^k. □

2.6 Complexes, K-Theory and Determinant Categories of Fréchet-Stein Algebras

For a Fréchet-Stein algebra A_∞ there are two abelian categories of modules which are interesting, firstly the category of modules over A_∞ and secondly the category of inverse systems of modules over $(A_n)_n$:

Definition 2.6.1. As usual we denote the derived category of the category of A_∞-modules by $\mathbf{D}(A_\infty)$ and the derived category of the category of inverse systems of $(A_n)_n$-modules by $\mathbf{D}_{\mathrm{sh}}(A_\infty)$.

The usual finiteness condition for A_∞-modules is too restrictive, hence:

Definition 2.6.2. We say that a complex of A_∞-modules is *globally perfect* if there exists a quasi-isomorphism to a bounded complex of coadmissable, projective A_∞-modules. We denote the full subcategory of globally perfect complexes by $\mathbf{D}_{\mathrm{coad.perf.}}(A_\infty)$.

The category of complexes of inverse systems of $(A_n)_n$-modules is equivalent to the category of inverse systems of complexes of A_n-modules, in particular there are projection operators $\mathbf{D}_{\mathrm{sh}}(A_\infty) \to \mathbf{D}(A_n)$.

Definition 2.6.3. We say that a complex of inverse systems $((M_n^i)_n)^i$ is *locally perfect* if for all n the projections to A_n-complexes are perfect, i.e. there exists a quasi-isomorphism

$$P_n^\bullet \to M_n^\bullet$$

where $P_n^\bullet$ is a perfect complex and the induced maps

$$A_n \otimes_{A_{n+1}} M_{n+1}^\bullet \to M_n^\bullet$$

are also quasi-isomorphisms.

We denote the full subcategory of locally perfect objects by $\mathbf{D}_{\mathrm{sh,perf}}(A_\infty)$.

Remark 2.6.4. A locally perfect complex does not have to be bounded. The subcategories of bounded complexes are defined in the obvious way.

Remark 2.6.5. There are the obvious maps

$$\mathbf{D}_{\mathrm{coad.perf.}}(A_\infty) \xrightarrow{(A_n \otimes^L_{A_\infty} -)_n} \mathbf{D}^b_{\mathrm{sh,perf}}(A_\infty),$$

note that $A_n \otimes_{A_\infty} -$ is exact, see corollary 2.4.8(i), and

$$\mathbf{D}^+_{\mathrm{sh,perf}}(A_\infty) \underset{\longrightarrow}{\overset{\mathbf{R}\varprojlim}{\longleftarrow}} \mathbf{D}(A_\infty).$$

For a K-Fréchet-Stein algebra A_∞ one has the usual determinant category $\mathbf{Det}(A_\infty)$ which just takes the finitely generated, projective modules into account. However the finiteness condition is again too restrictive, hence:

Definition 2.6.6. Let $\mathrm{PMod}^{\mathrm{coad}}(A_\infty)$ be the full subcategory of the abelian category of coadmissable modules which consists of the objects which are locally projective.

Lemma 2.6.7. *The category* $\mathrm{PMod}^{\mathrm{coad}}(A_\infty)$ *is exact.*

Proof. According to [Qui75, p. 91] and corollary 2.4.8(iii) we just have to check that the full subcategory $\mathrm{PMod}^{\mathrm{coad}}(A_\infty)$ is closed under extensions, i.e. assume that

$$0 \longrightarrow P_\infty \longrightarrow M_\infty \longrightarrow P'_\infty \longrightarrow 0$$

is a short exact sequence of coadmissable modules where the modules P_∞ and P'_∞ are locally projective. Then we have to show that M_∞ is also locally projective. However the functor $A_n \otimes_{A_\infty} -$ is exact (see corollary 2.4.8(i)), hence for every n

$$0 \longrightarrow P_n \longrightarrow M_n \longrightarrow P'_n \longrightarrow 0$$

is also exact. Since P'_n is projective, the short exact sequence splits. The lemma follows. □

Now, using this category we are able to define a K-theory and a determinant category:

Definition 2.6.8. Set

$$K_i^{\mathrm{coad}}(A_\infty) := K_i(\mathrm{PMod}^{\mathrm{coad}}(A_\infty))$$

and

$$\mathbf{Det}^{\mathrm{coad}}(A_\infty) := \mathbf{Det}(\mathrm{PMod}^{\mathrm{coad}}(A_\infty)).$$

However, for our applications the definition is still too restrictive, hence we introduce the following K-theory groups and the following determinant category:

Definition 2.6.9. Set

$$K_i^{\mathrm{sh}}(A_\infty) \coloneqq \varprojlim_n K_i(A_n)$$

where the canonical maps $K_i(A_{n+1}) \to K_i(A_n)$ are induced by $A_{n+1} \to A_n$ and

$$\mathbf{Det}^{\mathrm{sh}}(A_\infty) \coloneqq \mathrm{QCohSh}(\mathbf{Det}(A_n)_n)$$

where $\mathrm{QCohSh}(\mathbf{Det}(A_n)_n)$ is the category of inverse systems $(X_n)_n$ with the condition

$$A_n \otimes_{A_{n+1}} X_{n+1} \xrightarrow{\sim} X_n.$$

A morphism of objects in $\mathbf{Det}^{\mathrm{sh}}(A_\infty)$ is a morphism of the underlying inverse system.

Only using this category we get:

Remark 2.6.10. Note that there is a natural determinant functor

$$\mathbf{d}_{A_\infty} : \quad (\mathbf{D}_{\mathrm{sh,perf}}(A_\infty), \mathrm{is}) \to \mathbf{Det}^{\mathrm{sh}}(A_\infty)$$

(see section 1.5).

Remark 2.6.11. One can wonder about the relation of $K_i^{\mathrm{coad}}(A_\infty)$ and $K_i^{\mathrm{sh}}(A_\infty)$. At least for Schneider-Teitelbaum's distribution algebras for compact p-adic Lie groups, which we will cover in the next chapter, we have the following statements in the literature. Tamás Csige's thesis [Csi16] suggests that the groups $K_0(A_n)$ are independent of n. Furthermore Zábrádi's theorem 3.5.5, which is recalled below, states that in this case coadmissable projective modules are finitely generated, hence we suspect that the two K_0-groups should be canonically isomorphic.

3 Non-Commutative Analytic Spaces

Here we extend the theory of Tate algebras and affinoids to a non-commutative setting and then consider the continuous group cohomology of modules over these K-algebras.

3.1 nc-Tate Algebras

Using definition 1.6.1, we define our analogue of the Tate algebra as follows:

Definition 3.1.1. The *nc-Tate algebra* T_n^{nc} is $K\{|X_1, \ldots, X_n|\}$ with the K-Banach norm

$$\left| \sum_{I \subset \{1,\ldots,n\}^*} c_I X^I \right|_{T_n^{nc}} := \max_{I \subset \{1,\ldots,n\}^*} |c_I|_K,$$

the maximum exists since $c_I \to 0$ by definition.

Remark 3.1.2. The nc-Tate algebra is obviously orthonormalisable in the sense of definition 2.3.2 as the $(X^I)_I$ form a Schauder basis.

Lemma 3.1.3. *The canonical map $T_m^{nc} \twoheadrightarrow T_n^{nc}$ has Banach norm 1 for all $m \geq n$.*

Proof. Denote the map by p_n^m. Then

$$\left| \sum_{I \subset \{1,\ldots,m\}^*} c_I X^I \right|_{T_m^{nc}} = \max_{I \subset \{1,\ldots,m\}^*} |c_I|_K$$

and

$$\left| p_n^m \left(\sum_{I \subset \{1,\ldots,m\}^*} c_I X^I \right) \right|_{T_m^{nc}} = \left| \sum_{I \subset \{1,\ldots,n\}^*} c_I X^I \right|_{T_n^{nc}} = \max_{I \subset \{1,\ldots,n\}^*} |c_I|_K.$$

Hence,

$$|p_n^m(x)|_{T_n^{nc}} \leq |x|_{T_m^{nc}}$$

for $x \in T_m^{nc}$. Since

$$|1|_{T_m^{nc}} = 1 = |1|_{T_n^{nc}} = |p_n^m(1)|_{T_n^{nc}}$$

the desired statement follows. □

The usual Tate algebra T_n is obviously a quotient of T_n^{nc}:

Lemma 3.1.4. *The canonical map $T_n^{nc} \twoheadrightarrow T_n$ has Banach norm 1.*

Proof. Denote the above map by p. We define the map $\alpha : \{1,\dots,m\}^* \to \mathbb{N}^m$ where the ith coordinate of $\alpha(I)$ counts the occurrences of the symbol i in the string I. Then

$$\begin{aligned}
\left| p\left(\sum_{I\subset\{1,\dots,m\}^*} c_I X^I \right) \right|_{T_n} &= \left| \sum_{J\in\mathbb{N}^m} \left(\sum_{\alpha(I)=J} c_I \right) X^J \right|_{T_n} \\
&= \max_J \left| \sum_{\alpha(I)=J} c_I \right|_K \\
&\le \max_I |c_I|_K \\
&\le \left| \sum_{I\subset\{1,\dots,m\}^*} c_I X^I \right|_{T_m^{nc}}
\end{aligned}$$

where the inequality holds due to the strong triangle inequality. □

3.2 nc-Affinoid Algebras

Then the definition of the non-commutative analogue of affinoids is obviously:

Definition 3.2.1. Let A be a topological K-algebra and assume that there is a continuous K-algebra surjection $\varphi : T_n^{nc} \twoheadrightarrow A$. Then A is called an *nc-affinoid* K-algebra if $\ker\varphi$ is closed in T_n^{nc} and $T_n^{nc}/\ker\varphi \cong A$ is a *homeomorphism*.

Lemma 3.2.2. *A structure morphism $T_n^{nc} \twoheadrightarrow A$ is open.*

Proof. It is surjective, hence the claim follows by the open mapping theorem 2.1.9. □

Lemma 3.2.3. *An nc-affinoid algebra is a K-Banach space.*

Proof. From general facts (see [BGR84, Prop. 2.1.2/1,3]) it follows that the quotient norm is a norm and that the induced quotient topology is again complete. □

Remark 3.2.4. Hence, there is an induced K-Banach algebra structure on A. Furthermore, lemma 3.1.3 implies that the notion of nc-affinoid is independent of n.

Lemma 3.2.5. *The usual Tate algebra T_n is an nc-affinoid. In particular, an affinoid algebra is an nc-affinoid algebra (with the same induced K-Banach algebra structure).*

Proof. Use lemma 3.1.4. □

Lemma 3.2.6. *Assume that A is a Hausdorff topological ring and that there is a continuous, surjective homomorphism $T_n^{nc} \to A$. Then there is a topology on A such that A equipped with this topology is an nc-affinoid. If A is a K-Banach algebra, A is an nc-affinoid using the original topology.*

Proof. We just have to observe that the kernel of the homomorphism is closed, hence we can choose the quotient topology on A. If A is a K-Banach algebra then the homomorphism is open by the open mapping theorem 2.1.9, thus the quotient topology on A coincides with the original one. □

Definition 3.2.7. In the context of the last lemma we say that the K-Banach quotient norm on A induced by the surjection $s : T_n^{nc} \to A$ is the *norm associated with s*. Note that even if A is a K-Banach algebra these norms might not coincide.

Lemma 3.2.8. *Let A be an nc-affinoid K-algebra and $(a_i)_i \in \mathcal{O}_A^n$ be integral elements. Then the map $s : T_n^{nc} \to A$ given by $s(\sum_J c_J X^J) = \sum_J c_J a^J$ is a well-defined, continuous morphism of nc-affinoids.*

Proof. Let $T_n^{nc} \to A$ be the structure morphism of A. As $\mathcal{O}_K\{|X_1, \dots, X_n|\}[\pi^{-1}] = K\{|X_1, \dots, X_n|\}$, it suffices to show that the map

$$\begin{aligned} s : \mathcal{O}_K\{|X_1, \dots, X_n|\} &\longrightarrow A \\ \sum_J c_J X^J &\longmapsto \sum_J c_J a^J \end{aligned}$$

is well-defined and continuous.

The sum $\sum_J c_J a^J$ converges as $c_J a^J$ converges to 0 since the $a_i \in \mathcal{O}_A$ are by definition bounded by 1 and c_J converges π-adically to 0. Hence $\sum_{\#J<n} c_J a^J$ gets arbitrarily close to $\sum_J c_J a^J$, i.e.

$$\left|\sum_J c_J a^J\right| = \lim_n \left|\sum_{\#J<n} c_J a^J\right| \leq \sup_n \max_{\#J<n} \{|c_J||a^J|\} \leq \sup_n \max_{\#J<n} \{|c_J|\} = \left|\sum_J c_J X^J\right|.$$

Hence, $|s| \leq 1$ and we deduce that s is continuous. □

3.3 Integral nc-Affinoid Algebras

Definition 3.3.1. We call a π-adically complete[1] $\mathcal{O}_K$-Banach-algebra $\mathcal{A}$ such that $\mathcal{A}/\langle\pi\rangle$ is an ncft-k-algebra[2] an *integral nc-affinoid $\mathcal{O}_K$-algebra.*

Lemma 3.3.2. *An $\mathcal{O}_K$-algebra morphism $f : \mathcal{A} \to \mathcal{B}$ of integral nc-affinoids is automatically continuous.*

Proof. It suffices to show that $f^{-1}(\pi^m\mathcal{B})$ is open. If $x \in f^{-1}(\pi^m\mathcal{B})$ then we have to find an open set around x contained in $f^{-1}(\pi^m\mathcal{B})$. The open set $x + \pi^m\mathcal{A}$ fulfils this requirement. □

Proposition 3.3.3. *For an integral nc-affinoid $\mathcal{O}_K$-algebra $\mathcal{A}$ the K-Banach algebra $A = \mathcal{A}[\pi^{-1}]$ equipped with the induced topology is an nc-affinoid K-algebra.*

Proof. We need to define a continuous, surjective morphism $T_n^{nc} \to A$ by lemma 3.2.6 as A is a K-Banach algebra. Hence we have to find a continuous, surjective morphism $\mathcal{O}_K\{|X_1, \dots, X_n|\} \to \mathcal{A}$. Note that any morphism $\mathcal{O}_K\{|X_1, \dots, X_n|\} \to \mathcal{A}$ is continuous by lemma 3.3.2.

Let $(\overline{x_i})_i$ be the generator of the ncft-k-algebra $\mathcal{A}/\langle\pi\rangle$ and let x_i be any lift of $\overline{x_i}$. We define the function

$$\begin{aligned} s : \mathcal{O}_K\{|X_1, \dots, X_n|\} &\to \mathcal{A} \\ \sum_J c_J X^J &\mapsto \sum_J c_J x^J \end{aligned}$$

with $c_J \to 0$ for $|J| \to \infty$. The map s is well-defined since $c_J x^J$ is a zero-sequence in the π-adic topology of $\mathcal{A}$ as c_J is a zero-sequence.

Because $\bar{s} : \mathcal{O}_K\{|X_1, \dots, X_n|\} \to \mathcal{A}/\langle\pi\rangle$ is surjective we deduce the proposition by our version of the I-adic Nakayama's lemma (see corollary 1.7.2) with $R = \mathcal{O}_K$, $I = \langle\pi\rangle$, $A = \mathcal{O}_K\{|X_1, \dots, X_n|\}$ and $B = \mathcal{A}$. □

3.4 Berthelot's Construction: Associated nc-Affinoid Algebras

We now generalise the constructions of [Jon95, §7.1.1] and [Pot13, §1.4] to the non-commutative setting and prove important standard results.

[1] in the sense of definition 2.1.17

[2] in the sense of definition 1.6.2

We are assuming in this section that Λ and I fulfil the following:

Hypothesis 3.4.1. Let Λ be an I-adically complete $\mathcal{O}_K$-algebra where I is a two-sided ideal of Λ which contains a power of π, say $\pi^\mu \in I$.

Definition 3.4.2. For every n we define $\Lambda_n^0 := \Lambda[I^n/\pi] \subset \Lambda[\pi^{-1}]$ as the sub-algebra (with 1) generated by $\lambda \in \Lambda$ and i/π with $i \in I^n$. Furthermore, we define Λ_n as the I_n^0-adic completion of Λ_n^0 with $I_n^0 := \langle I \rangle$ and set $I_n = \langle I_n^0 \rangle \subset \Lambda_n$. Lastly, we define $A_n = \Lambda_n[\pi^{-1}]$.

Lemma 3.4.3. *Assume Λ fulfils hypothesis 3.4.1 and is left noetherian. Furthermore we require Λ/I^m to be an ncft-$\mathcal{O}_K/\langle \pi^{\mu m} \rangle$-algebra for all m. Then Λ_n/I_n^m is a discrete ncft-$\mathcal{O}_K/\langle \pi^{\mu m} \rangle$-algebra for all $n, m \geq 1$.*

Proof. We first note that $\Lambda_n^0 = \sum_i \frac{I^{ni}}{\pi^i} \subset \Lambda[\pi^{-1}]$, hence

$$\langle I^m \rangle = \Lambda_n^0 I^m \Lambda_n^0 = \left(\sum_i \frac{I^{ni}}{\pi^i} \right) I^m \left(\sum_i \frac{I^{ni}}{\pi^i} \right) = \sum_i \frac{I^{ni+m}}{\pi^i}$$

and

$$(I_n^0)^m = \langle I \rangle^m = \left(\sum_i \frac{I^{ni+1}}{\pi^i} \right)^m = \sum_i \frac{I^{ni+m}}{\pi^i}$$

are equal. Because Λ is left noetherian we can find left generators $x_1, \ldots, x_a$ of I^n. Hence the $\mathcal{O}_K$-algebra morphism

$$\begin{aligned} \Lambda\langle X_1, \ldots, X_a \rangle_{\mathcal{O}_K\text{-nc}} &\longrightarrow \Lambda_n^0 = \sum_i \frac{I^{ni}}{\pi^i} \\ X_j &\longmapsto \frac{x_j}{\pi} \end{aligned}$$

is surjective where the $\mathcal{O}_K$-algebra $\Lambda\langle X_1, \ldots, X_a \rangle_{\mathcal{O}_K\text{-nc}}$ is the $\mathcal{O}_K$-algebra

$$\mathcal{O}_K\langle [\lambda]\, (\lambda \in \Lambda), X_1, \ldots X_a \rangle / \sim$$

where the relation $\sim$ is generated by:

$$\begin{aligned} \alpha[\lambda_1] &\sim [\alpha\lambda_1] \\ [\lambda_1] + [\lambda_2] &\sim [\lambda_1 + \lambda_2] \qquad \text{and} \\ [\lambda_1] \cdot [\lambda_2] &\sim [\lambda_1 \cdot \lambda_2] \end{aligned}$$

for $\lambda_i \in \Lambda$ and $\alpha \in K$. I.e. the X_j do *not* commute in general with elements in Λ but we have that $\mathcal{O}_K$ is in the centre of $\Lambda\langle X_j\rangle_{\mathcal{O}_K\text{-nc}}$. Then

$$(\Lambda/I^m)\,\langle X_1,\ldots,X_a\rangle_{\mathcal{O}_K\text{-nc}} \longrightarrow \Lambda_n^0/\langle I^m\rangle = \Lambda_n^0/(I_n^0)^m = \Lambda_n/I_n^m$$

is also surjective. Furthermore we know that Λ/I^m is an ncft-$\mathcal{O}_K/\langle\pi^{\mu m}\rangle$-algebra, i.e. there is a surjection

$$(\mathcal{O}_K/\langle\pi^{\mu m}\rangle)\,\langle Y_i\rangle \twoheadrightarrow \Lambda/I^m$$

thus we find that the composition of the $\mathcal{O}_K$-algebra morphisms

$$(\mathcal{O}_K/\langle\pi^{\mu m}\rangle)\,\langle Y_i,X_j\rangle = \big(\,(\mathcal{O}_K/\langle\pi^{\mu m}\rangle)\,\langle Y_i\rangle\big)\langle X_j\rangle_{\mathcal{O}_K\text{-nc}} \twoheadrightarrow (\Lambda/I^m)\,\langle X_j\rangle_{\mathcal{O}_K\text{-nc}} \twoheadrightarrow \Lambda_n/I_n^m$$

is also a surjection. □

Theorem 3.4.4. *For a left noetherian Λ as in hypothesis 3.4.1 such that Λ/I^m is an ncft-$\mathcal{O}_K/\langle\pi^{\mu m}\rangle$-algebra for all m, Λ_n is an integral nc-affinoid $\mathcal{O}_K$-algebra for all n. In particular $A_n = \Lambda_n[\pi^{-1}]$ is an nc-affinoid K-algebra.*

Proof. In $\Lambda_n^0 = \Lambda[I^n/\pi]$ we find $\langle I^n\rangle \subset \langle\pi\rangle$ and $\langle\pi^\mu\rangle \subset \langle I\rangle$ as two-sided ideals. Hence Λ_n is also the π-adic completion of Λ_n^0 and $\Lambda_n/\langle\pi\rangle = \Lambda_n^0/\langle\pi\rangle$ is an ncft-k-algebra because Λ_n^0/I_n^n is an ncft-$\mathcal{O}_K/\langle\pi^{\mu n}\rangle$-algebra by lemma 3.4.3. □

Lemma 3.4.5. *Assume that (Λ, I) and (Λ', I') fulfil the conditions of the previous theorem. Then a continuous morphism $f : \Lambda \to \Lambda'$ induces an $\mathcal{O}_K$-algebra morphism $f_n : \Lambda_{m(n)} \to \Lambda'_n$ of integral nc-affinoids for all n where $m(n)$ is an integer such that $f^{-1}((I')^n)$ contains $I^{m(n)}$.*

Proof. The existence of the morphism is obvious due to the existence of $m(n)$. □

Lemma 3.4.6. *Let M be a left K-Banach space equipped with a compatible left B-multiplication, where B is a K-Banach algebra, i.e. norm of $B \to \mathrm{End}_{K,\mathrm{cts}}(M)$, a map of K-Banach spaces, should be 1. Assume that we have a continuous right action of Λ on M, Λ as in hypothesis 3.4.1, which commutes with the B-multiplication on M and comes from an action on $\mathcal{O}_M$. Then there exists an n_0 such that the action of Λ on M naturally factors through A_n for all $n \geq n_0$.*

Proof. Note that we only consider continuous homomorphisms, hence we drop the index "cts" to ease the notation.

By assumption there is a ring homomorphism

$$\varphi : \quad \Lambda^{\mathrm{op}} \longrightarrow \mathcal{O}_{\mathrm{End}_B(M)} = \mathrm{End}_{\mathcal{O}_B}(\mathcal{O}_M) \subset \mathrm{End}_B(M)$$

which is continuous. Hence, $\varphi^{-1}(\pi \, \mathrm{End}_{\mathcal{O}_B}(\mathcal{O}_M))$ is open in Λ, which implies by the definition of the I-adic topology the existence of an $n_0 \gg 0$ such that $I^n \subset \varphi^{-1}(\pi \, \mathrm{End}_{\mathcal{O}_B}(\mathcal{O}_M))$ for $n \geq n_0$, i.e. $\varphi(I^n) \subset \pi \, \mathrm{End}_{\mathcal{O}_B}(\mathcal{O}_M)$. Hence, the induced morphism

$$\varphi : \quad (\Lambda_n^0)^{\mathrm{op}} = \Lambda[I^n/\pi]^{\mathrm{op}} \to \mathrm{End}_{\mathcal{O}_B}(\mathcal{O}_M)$$

is well-defined and also extends to the π-adic completion as well as the localisation at π of both sides.

Finally we deduce the commutativity of

$$\begin{tikzcd}
\Lambda^{\mathrm{op}} \arrow[r] \arrow[dr] & A_n^{\mathrm{op}} \arrow[d] & A_{n+1}^{\mathrm{op}} \arrow[l] \arrow[dl] & A_{n+2}^{\mathrm{op}} \arrow[l] \arrow[dll] & \cdots \arrow[l] \arrow[dlll] \\
 & \mathrm{End}_B(M) & & &
\end{tikzcd}$$

for $n \geq n_0$ which in fact holds because

$$\begin{tikzcd}
\Lambda^{\mathrm{op}} \arrow[r] \arrow[d] & (\Lambda_n^0)^{\mathrm{op}} \arrow[d] & (\Lambda_{n+1}^0)^{\mathrm{op}} \arrow[l] \arrow[dl] & (\Lambda_{n+2}^0)^{\mathrm{op}} \arrow[l] \arrow[dll] & \cdots \arrow[l] \arrow[dlll] \\
\mathrm{End}_B(M) & \Lambda[1/\pi]^{\mathrm{op}} \arrow[l] & & &
\end{tikzcd}$$

obviously commutes. □

3.5 Associated Fréchet-Stein Algebras

Definition 3.5.1. Let A_n be an nc-affinoid associated to Λ as in definition 3.4.2. We call $A_\infty := \varprojlim A_n$ for $n \geq 2$ the *associated Fréchet-Stein algebra to* Λ if it is indeed a two-sided Fréchet-Stein algebra. We equip A_∞ with the projective limit topology.

We next pose the following question: when does the associated Fréchet-Stein algebra exist? The general answer seems to be complicated. However in the context of most interest to us, e.g. the Iwasawa algebra of Lie groups like $\mathrm{GL}_2(\mathbb{Z}_p)$, we now interpret

Schneider-Teitelbaum's fundamental theorem in the language of the previous section:

Theorem 3.5.2. *Assume that G is a compact p-adic Lie group and Λ is the completed group algebra $\mathcal{O}_K[[G]]$. Then with the notation of definition 3.4.2:*

(i) Λ, $\operatorname{gr}_I \Lambda$, Λ_n and A_n are left and right noetherian for $n \geq 2$,

(ii) A_n is flat as a left and right A_{n+1}-module,

(iii) $A_\infty := \varprojlim A_n$ is a two-sided K-Fréchet-Stein algebra,

(iv) let q_n be the semi-norm induced by the projection map $A_\infty \to A_n$, then $(A_\infty)_{q_n}$ as defined in 2.4.1 is naturally isomorphic to A_n, i.e. the notation of section 2.4 is consistent with this section, and

(v) $\Lambda[\pi^{-1}] \longrightarrow A_\infty$ is faithfully flat.

Proof. We state the proof of this theorem for $p > 2$, the case $p = 2$ follows after minor obvious modifications.

The proof of the theorem depends on the next proposition which states that for $n \geq 2$ and $p > 2$ there is a norm-preserving isomorphism

$$\alpha_n : \quad A_n \xrightarrow{\sim} D_{p^{-1/n}}(G, \mathbb{Q}_p)$$

where the completed distribution algebra $D_r(G, \mathbb{Q}_p)$ is defined in [ST03, Thm. 5.1].

We can immediately restrict ourselves to the case of $K = \mathbb{Q}_p$ as K in general is a finite extension of $\mathbb{Q}_p$ and tensoring with finite free modules does not affect completeness, fully faithfulness and the noetherian property.

Due to the above identification we see that the last part of (i), (ii) and (iii) is just [ST03, Thm. 5.1] (we are in the case $L = \mathbb{Q}_p$) and (v) is [ST03, Thm. 5.2].

Furthermore (iv) follows since

$$D(G, \mathbb{Q}_p) \xrightarrow{\sim} \varprojlim D_r(G, \mathbb{Q}_p)$$

holds due to [ST03, p. 151].

The norm-preserving isomorphism α_n above identifies Λ_n with $F^0 D_{r_n}(G, K)$ which is canonically isomorphic to $F^0 D_{r_n}(H, K)^d$. Thus, in order to prove the third part of (i) it therefore suffices to show that $F^0 D_{r_n}(H, K)$ is noetherian. We would like to apply [ST03, Prop. 1.1], i.e. we need to show that $\operatorname{gr}^\bullet_{r_n} F^0 D_{r_n}(H, K)$ is noetherian. This ring however is isomorphic to $\operatorname{gr}^\bullet_{r_n} \Lambda(H)$, which is noetherian by [ST03, Thm. 4.5(ii)].

Similarly the first part of (i) follows from [ST03, Thm. 4.5(ii)] which shows that $\Lambda(H)$ is noetherian.

We will see below $\langle I\rangle^n = \sum I^n g_i$ where $g_1,\dots,g_a$ are the coset representatives of H in G. Hence we also find $\operatorname{gr}_{\langle I\rangle}\Lambda(G) = \oplus \operatorname{gr}_I \Lambda(H)\overline{g}_i$ with $\overline{g}_i \in \operatorname{gr}^0_{\langle I\rangle}$. Therefore the second part of (i) also holds because $\operatorname{gr}_I \Lambda(H)$ is noetherian by [ST03, Rem. 4.6]. □

Example 3.5.3. We see that example 2.4.3 is just the previous situation for $G = \mathbb{Z}_p$.

We now identify our A_n with Schneider-Teitelbaum's completed distribution algebra $D_{p^{-1/n}}(G,\mathbb{Q}_p)$:

Proposition 3.5.4. *For $p > 2$ there is a canonical norm-preserving isomorphism*

$$\alpha_n : \quad A_n \xrightarrow{\sim} D_{p^{-1/n}}(G,\mathbb{Q}_p)$$

for $n \geq 2$ of $\mathbb{Z}_p[[G]]$-algebras which is natural in n, where $D_r(G,\mathbb{Q}_p)$ is defined in [ST03, Thm. 5.1]. If $p = 2$, the right hand side has to be replaced by $D_{p^{-1/2n}}(G,\mathbb{Q}_p)$.

Proof. We now recall some notation and facts from [ST03, §4,5] however we will not review all the results of these section, i.e. it is assumed that the reader is familiar with the work of Schneider and Teitelbaum.

G has an open normal subgroup H which is a uniform pro-p-group. According to [DDMS, §4.2] there is an ordered basis $h_1,\dots,h_d$ of H such that $\omega(h_i) = 1$ for $p > 2$ respectively $\omega(h_i) = 2$ for $p = 2$ (see also [DS13, §2.4]). We will first discuss the case $p > 2$ and afterwards we discuss the necessary modifications for $p = 2$.

The assumption implies that $\tau\alpha$ is just $|\alpha|$ (see [ST03, p. 160]). Set $b_i := h_i - 1 \in \mathbb{Z}_p[[G]]$ and we define $I = (p, b_1,\dots,b_d) \subset \mathbb{Z}_p[[H]]$ according to [FK06, §1.4.2]. For this proof we agree on the following convention: $\Lambda(G)^0_n$, $\Lambda(G)_n$ and $A(G)_n$ are the objects which were associated with $\Lambda(G) := \mathbb{Z}_p[[G]]$ in the last section using $\langle I\rangle \subset \Lambda(G)$. We also let r_n be $p^{-1/n}$. Elements in $D_{r_n}(H,\mathbb{Q}_p)$ can be uniquely represented by a power series

$$\sum_\alpha d_\alpha b^\alpha \quad \text{with } d_\alpha \in K \text{ and } |d_\alpha|_p r_n^{|\alpha|} \to 0$$

and we define

$$\left|\sum_\alpha d_\alpha b^\alpha\right|_{r_n} := \max |d_\alpha|_p r_n^{|\alpha|}.$$

The image of the canonical embedding $\Lambda(H) \hookrightarrow D_{r_n}(H,\mathbb{Q}_p)$ corresponds to all the elements with $d_\alpha \in \mathbb{Z}_p$ (see [ST03, p. 160]).

$\Lambda(H)_n^0$ embedded in $D_{r_n}(H,\mathbb{Q}_p)$: We claim that $\Lambda(H)_n^0 = \Lambda(H)[I^n/p]$ embedded in $D_{r_n}(H,\mathbb{Q}_p)$ can be written as

$$X = \left\{ x = \sum_\alpha d_\alpha b^\alpha \,\middle|\, |x|_{r_n} \leq 1 \text{ and } |d_\alpha|_p \text{ is bounded above} \right\}.$$

We can see the inclusion of the image of $\Lambda(H)[I^n/p]$ in X as follows: we have $\Lambda(H)[I^n/p] = \sum I^{ni}/p^i$, the boundedness of $|d_\alpha|_p$ is clear and it suffices to check that $|I^{ni}/p^i|_{r_n} \leq 1$ and because the norm is submultiplicative (see [ST03, Prop. 4.2]), it suffices to check the inequality for $i = 0, 1$. It is clear for $i = 0$ as $\Lambda(H)$ corresponds to the elements with coefficients in $\mathbb{Z}_p$. Regarding $i = 1$ we have:

$$\left| \frac{I^n}{p} \right|_{r_n} \leq p\,|I|_{r_n}^n \leq \max p\,|b_i|_{r_n}^n = p r_n^n = p \cdot \left(p^{-1/n}\right)^n = 1.$$

On the other hand X is included in the image: we first show that for every α the element $d_\alpha b^\alpha \in D_{r_n}(H,\mathbb{Q}_p)$ lies in $\Lambda(H)[I^n/p]$. As $|d_\alpha|_p r_n^{|\alpha|} \leq 1$ holds, we can deduce that $|\alpha| \geq -n\nu_p(d_\alpha)$. We can assume that $\nu_p(d_\alpha)$ is negative otherwise there is nothing to show. Then

$$d_\alpha b^\alpha = \left(p^{|\nu_p(d_\alpha)|} d_\alpha\right) \cdot \left(\frac{b^\alpha}{p^{|\nu_p(d_\alpha)|}}\right)$$

where the first factor is in $\mathbb{Z}_p$ and the second factor is in $I^{|\alpha|}/p^{|\nu_p(d_\alpha)|} \subseteq I^{n|\nu_p(d_\alpha)|}/p^{|\nu_p(d_\alpha)|}$ which shows $d_\alpha b^\alpha \in \Lambda(H)[I^n/p]$.

Let us assume that the upper bound $|d_\alpha|_p$ is p^c. Decompose $x \in X$ into $x_1 + x_2$ such that x_2 has all the terms with $|\alpha| \geq n \cdot c$. In particular x_1 consists of finitely many summands, hence x_1 lies in the image by the reasoning above. Regarding x_2 we would like to show $x_2 \in I^{nc}/p^c$, i.e. $p^c x_2 \in I^{nc}$. Indeed:

$$p^c x_2 = \sum_{|\alpha| \geq nc} p^c d_\alpha b^\alpha = \sum_{|\alpha| \geq nc} (p^c d_\alpha)\, b^{\alpha_1} b^{\alpha_2} = \sum_{|\alpha_2| = nc} \left(\sum_{|\alpha_1| \geq 0} (p^c d_\alpha)\, b^{\alpha_1} \right) b^{\alpha_2}$$

where we decompose α into $\alpha_1 + \alpha_2$ such that $b^\alpha = b^{\alpha_1} b^{\alpha_2}$ holds and in the second sum we only allow α_1 which fulfil this requirement. The desired claim follows as there are only finitely many α_2 with $|\alpha_2| = nc$, and the inner sum is in the image of $\Lambda(H)$ since the coefficients are in $\mathbb{Z}_p$ by the choice of c.

$A(H)_n$ vs $D_{r_n}(H,\mathbb{Q}_p)$: A similar reasoning as above yields that the image of $(I_n^0)^m = \langle I^m \rangle \subset \Lambda(H)_n^0$ in $D_{r_n}(H,\mathbb{Q}_p)$ are precisely the elements with $|x|_{r_n} \leq p^{-m}$ and $|d_\alpha|_p$ is bounded above.

Hence, the left and right $\Lambda(H)$-morphism

$$\alpha_n^{H,m}: \quad \Lambda(H)_n^0/(I_n^0)^m \to F^0 D_{r_n}(H,\mathbb{Q}_p)/F^m D_{r_n}(H,\mathbb{Q}_p)$$

has trivial kernel as $F^m D_{r_n}(H,\mathbb{Q}_p) \cap \Lambda(H)_n^0 = (I_n^0)^m$ holds using the characterisations of $\Lambda(H)_n^0$ and $(I_n^0)^m$ as above (regarding the definition of the filtration see [ST03, p. 162]). Furthermore, an element x of $F^0 D_{r_n}(H,\mathbb{Q}_p)$ can be written as x_1+x_2 where x_1 is a finite sum and $|x_2|_{r_n} \leq p^{-m}$ using the power series representation expansion. As $x_2 \in F^m D_{r_n}(H,\mathbb{Q}_p)$ and x_1 is in the image of $\Lambda(H)_n^0$ we see that $\alpha_n^{H,m}$ is actually an isomorphism. By taking the projective limit we get the left and right $\Lambda(H)$-isomorphism $\alpha_n^H : \Lambda(H)_n \xrightarrow{\sim} F^0 D_{r_n}(H,\mathbb{Q}_p)$ which can be extended to the localisations:

$$\alpha_n^H: \quad A(H)_n \xrightarrow{\sim} D_{r_n}(H,\mathbb{Q}_p).$$

***I* vs *I^g*:** For $x \in D_{r_n}(H,\mathbb{Q}_p)$ and $g \in G$ we denote the conjugation action $\delta_g x \delta_{g^{-1}} \in D_{r_n}(H,\mathbb{Q}_p)$ of g on x by x^g, here δ_g is the Dirac δ distribution. By the discussion in [ST03, Thm. 5.1] we see that $(h_i^g)_i$ is also an ordered basis with $\omega(h_i^g) = \omega(h_i) = 1$. Using the norm $|{-}|_{1/p}$ on $\Lambda(G)$ induced by $\Lambda(G) \subset D(G,\mathbb{Q}_p)$ we see that I is exactly the subset with norm $\leq p^{-1}$. Furthermore, the norm is independent from the ordered basis (see [ST03, p. 160]) hence $I = I^g$.

Definition of α_n: Let $g_1, \ldots, g_a$ be a set of coset representatives of H in G. Then the map

$$\begin{aligned} \varphi: \quad \oplus D_{r_n}(H,\mathbb{Q}_p) &\longrightarrow D_{r_n}(G,\mathbb{Q}_p) \\ (x_i) &\longmapsto \sum x_i \delta_{g_i} \end{aligned}$$

is an isomorphism and the norm on $D_{r_n}(G,\mathbb{Q}_p)$ is defined to be the maximum norm via the left hand side (see the proof of [ST03, Thm. 5.1]). Furthermore,

$$\begin{aligned} \varphi: \quad \oplus \Lambda(H) &\longrightarrow \Lambda(G) \\ (x_i) &\longmapsto \sum x_i g_i \end{aligned}$$

is also an isomorphism. We note the following consequence:

$$\begin{aligned}\langle I\rangle = \Lambda(G)I\Lambda(G) &= \left(\sum_i \Lambda(H)g_i\right) I \left(\sum_j \Lambda(H)g_j\right)\\ &= \sum_{i,j}\Lambda(H)g_iIg_j = \sum_{i,j}\Lambda(H)g_iIg_j = \sum_{i,j}\Lambda(H)g_iIg_i^{-1}g_ig_j\\ &\subseteq \sum_{i,j}\Lambda(H)I^{g_i}g_{k(i,j)} = \sum_{i,j}\Lambda(H)Ig_{k(i,j)} = \sum_{i,j}Ig_{k(i,j)} = \sum_k Ig_k\end{aligned}$$

where $g_{k(i,j)}$ is the coset representative of the coset associated with g_ig_j and we used the fact that I is invariant under conjugation. As $\sum Ig_i \subseteq \langle I\rangle$ we have actually shown equality. Furthermore,

$$\begin{aligned}\langle I^m\rangle\langle I\rangle &= \sum_{i,j} I^m g_iIg_j = \sum_{i,j} I^m g_iIg_i^{-1}g_ig_j\\ &\subseteq \sum_{i,j} I^m I^{g_i}g_{k(i,j)} = \sum_k I^{m+1}g_k\end{aligned}$$

and again we can deduce that $\langle I^m\rangle\langle I\rangle = \sum I^{m+1}g_i$ which is also equal to $\langle I^{m+1}\rangle$. An easy induction argument shows that φ induces a left $\Lambda(H)$-module isomorphism of $\oplus I^m \subset \oplus\Lambda(H)$ and $\langle I^m\rangle = \langle I\rangle^m \subset \Lambda(G)$. Due to the equality

$$\Lambda(G)\left[\frac{\langle I^n\rangle}{p}\right] = \sum \frac{\langle I^{nj}\rangle}{p^j} = \sum \frac{I^{nj}}{p^j}g_i$$

we deduce that the induced morphism

$$\varphi: \quad \oplus\Lambda(H)\left[\frac{I^n}{p}\right] \longrightarrow \Lambda(G)\left[\frac{\langle I^n\rangle}{p}\right]$$

is an isomorphism of left $\Lambda(H)_n^0$-modules and the maximum norm on the left corresponds to the $\langle I\rangle_n^0$-norm on the right. Hence, φ induces an isomorphism on their respective completions

$$\varphi: \quad \oplus\Lambda(H)_n \xrightarrow{\sim} \Lambda(G)_n$$

and also on the localisation at p, i.e.

$$\varphi: \quad \oplus A(H)_n \xrightarrow{\sim} A(G)_n.$$

We note that the last isomorphism is a canonical, norm-preserving left $\Lambda(H)_n$-isomorphism which induces the isomorphism α_n in the commutative diagram

$$\begin{tikzcd}
A(G)_n \arrow[r, dashed, "\alpha_n", "\sim"'] & D_{r_n}(G, \mathbb{Q}_p) \\
\oplus A(H)_n \arrow[u, "\varphi", "\wr"'] \arrow[r, "\oplus\alpha_n^H", "\sim"'] & \oplus D_{r_n}(H, \mathbb{Q}_p) \arrow[u, "\wr"']
\end{tikzcd}$$

with the same properties where the right vertical arrow is defined in [ST03, Thm. 5.1]. We note that the same strategy shows that α_n is also a right $\Lambda(H)$-isomorphism.

α_n **is a** $\Lambda(G)$**-algebra homomorphism:** we would like to show that α_n is not only a left and right $\Lambda(H)$-module morphism but also a left and right $\Lambda(G)$-algebra morphism. As the both arguments are symmetric, we only present the left $\Lambda(G)$-algebra morphism property.

We first show that the embeddings of $\Lambda(G)$ in $D_{r_n}(G, \mathbb{Q}_p)$ and $A(G)_n$ commute i.e. the right triangle in the following diagram commutes:

$$\begin{tikzcd}
 & \oplus A(H)_n \arrow[rr, "\varphi", "\sim"'] \arrow[dd, "\oplus\alpha_n^H", "\wr"'] & & A(G)_n \arrow[dd, "\alpha_n", "\wr"'] \\
\oplus\Lambda(H) \arrow[ur, hook] \arrow[dr, hook] \arrow[rr, "\sim"] & & \Lambda(G) \arrow[ur, hook] \arrow[dr, hook] & \\
 & \oplus D_{r_n}(H, \mathbb{Q}_p) \arrow[rr, "\sim"] & & D_{r_n}(G, \mathbb{Q}_p).
\end{tikzcd}$$

As all the quadrilaterals commute it suffices to show the left triangle commutes. But this is clear as α_n is a left and right $\Lambda(H)$-module homomorphism. Now we can deduce that the restriction of α_n to $\Lambda(G)$ is multiplicative because the embeddings of $\Lambda(G)$ in $D_{r_n}(G, \mathbb{Q}_p)$ and $A(G)_n$ are multiplicative (see [ST03, p. 163]). Furthermore, note that $\Lambda(G)[p^{-1}]$ is dense in $D_{r_n}(G, \mathbb{Q}_p)$ and $A(G)_n$. Because the multiplication and α_n are continuous, we see that the multiplicative property extends to the full algebra, hence α_n is a $\Lambda(G)$-algebra homomorphism.

Naturality in n**:** We want to show that the diagram

$$\begin{tikzcd}
A(G)_{n+1} \arrow[r] \arrow[d, "\alpha_{n+1}", "\wr"'] & A(G)_n \arrow[d, "\alpha_n", "\wr"'] \\
D_{r_{n+1}}(G, \mathbb{Q}_p) \arrow[r] & D_{r_n}(G, \mathbb{Q}_p)
\end{tikzcd}$$

commutes. Using the continuity argument from above we can reduce the question to the following: is this a diagram commutative over $\Lambda(G)$? However this follows directly from the last commutative diagram.

For $p = 2$ the only modification is that $\tau\alpha$ is $2|\alpha|$, hence $A(G)_n$ corresponds to $D_{\sqrt{r_n}}(G, \mathbb{Q}_p)$. □

In the situation of theorem 3.5.2 all three notions of projective objects, which were discussed in section 2.5, coincide (see also remark 2.5.1).

Theorem 3.5.5 ([Záb12, Thm. 3.10])**.** *Let A_∞ be an associated Fréchet-Stein algebra as in theorem 3.5.2. Then a projective object in the category of coadmissable modules is finitely generated, hence projective.*

3.6 Continuous Group Cohomology

This section follows the development of [Pot13, §1] rather closely. Our goal is to understand group cohomology over non-commutative analytic spaces.

Definition 3.6.1. A topological $\mathcal{O}_K$-module M is said to be *linearly topologised* if 0_M has a basis of open neighbourhoods consisting of a decreasing sequence of $\mathcal{O}_K$-modules M_n. A topological $\mathcal{O}_K$-algebra A is said to be *linearly topologised* if it is linearly topologised as a module and the sub-modules R_n can be chosen such that $R_n \cdot R_n \subset R_n$. We say an R-module is *linearly topologised compatibly with* R if the systems R_n and M_n can be choosen such that $R_n \cdot M_n \subset M_n$.

Remark 3.6.2. Throughout the section we assume that the occurring $\mathcal{O}_K$-algebras and $\mathcal{O}_K$-modules are linearly topologised (in a compatible way).

Definition 3.6.3. Let G be a profinite group, R a topological ring and let M be a topological R-module with a continuous R-action and we additionally assume that M is a continuous $R[G]$-module, i.e. the group homomorphism $G \to \mathrm{Aut}_{R,\mathrm{cts}}(M)$ is continuous.[3] Then we define *the continuous cochain complex* $C^i_{\mathrm{cts}}(G, M)$ to be the continuous maps $\mathrm{Map}_{\mathrm{cts}}(G^i, M)$ with the standard differentials like in [NSW08, §2.7]. We set $\mathbf{R}\Gamma_{\mathrm{cts}}(G, M)$ to be the image of the complex $C^\bullet_{\mathrm{cts}}(G, M)$ in the derived category $\mathbf{D}^+(R)$.

We assume the following throughout this section to get a reasonable behaviour of group cohomology:

[3] We remind the reader that we topologise $\mathrm{Aut}_{R,\mathrm{cts}}(M)$ using the compact open topology.

Hypothesis 3.6.4. (i) Λ is an I-adically complete $\mathcal{O}_K$-algebra as required in set-up 3.4.1 such that Λ and $\operatorname{gr}_I \Lambda$ are left and right *noetherian* and I^n has the *right Artin-Rees property* for all n,

(ii) the profinite group G has finite p-cohomological dimension e and $H^i_{\mathrm{cts}}(G, M)$ is finite for all finite discrete $\mathbb{F}_p[G]$-modules M and

(iii) T is a projective, finitely generated (left) Λ-module equipped with the I-adic topology and a continuous $\Lambda[G]$-module structure.

Remark 3.6.5. Pottharst treats flat and finitely generated modules in [Pot13, §1.1]. Note however that his setup is not more general than ours as finitely generated implies finitely presented as Λ is noetherian and a finitely presented module which is also flat is automatically projective by [Wei94, Thm. 3.2.7].

Remark 3.6.6. We say that $\Lambda[Ix] = \Lambda + Ix + I^2x^2 + \ldots$ is the *Rees ring* associated with Λ and I (see [GW04, p. 223]). Assuming $\Lambda[Ix]$ is left (right) noetherian, we see that the quotients $\Lambda[Ix]/\langle x\rangle \xleftarrow{\sim} \Lambda$ and $\Lambda[Ix]/\langle I\rangle \xleftarrow{\sim} \operatorname{gr}_I \Lambda$ are left (right) noetherian. Furthermore, I has the left (right) Artin-Rees property by [GW04, thm. 13.2].

Hence, to show that Λ satisfies (i) we just have to check that the Rees rings $\Lambda[I^n x]$ are left and right noetherian.

Remark 3.6.7. We would also like to draw the attention to a fact which even though we cannot utilise at the moment is nevertheless interesting. Looking back at definition 3.4.2 we recognise that the Rees ring $\Lambda[I^n x]$ surjects onto Λ_n^0.

3.6.1 Preliminaries

Lemma 3.6.8. *Let X, Y be complete, right and left Λ-modules respectively equipped with the I-adic topology and assume that X is finitely generated. Then the natural map*

$$X \otimes_\Lambda Y \xrightarrow{\sim} X \widehat{\otimes}_\Lambda Y = \varprojlim \, (X \otimes_\Lambda Y) \,/\, (X \otimes_\Lambda I^n Y)$$

is an isomorphism.

Proof. We first note that X is automatically finitely presented as Λ is right noetherian. We deduce

$$X \otimes_\Lambda Y \xrightarrow{\sim} X \otimes_\Lambda \varprojlim Y/I^n Y \xrightarrow{\sim} \varprojlim \, (X \otimes_\Lambda Y) \,/\, (X \otimes_\Lambda I^n Y)$$

where the second isomorphism can be deduced by observing that $X \otimes_\Lambda -$ for X finitely presented commutes with projective limits of surjective systems. $\square$

Lemma 3.6.9. *Let M fulfil hypothesis 3.6.4(iii) except possibly the projectivity assumption. Then the natural map*

$$C^{\bullet}_{\mathrm{cts}}(G,M)/I^n \to C^{\bullet}_{\mathrm{cts}}(G,M/I^n)$$

is an isomorphism of complexes.

Proof. The proof is a verbatim copy of the proof of [Pot13, Lem. 1.3(2)]. □

Corollary 3.6.10. *The natural map*

$$C^{\bullet}_{\mathrm{cts}}(G,M) \to \varprojlim C^{\bullet}_{\mathrm{cts}}(G,M/I^n)$$

is an isomorphism of complexes.

Proof. This is precisely the statement of the universal property of inverse limits. □

Corollary 3.6.11. *Let N be a complete right Λ-module equipped with the I-adic topology. Then the natural map*

$$N \mathbin{\widehat{\otimes}_{\Lambda}} C^{\bullet}_{\mathrm{cts}}(G,M) \to C^{\bullet}_{\mathrm{cts}}(G,N \otimes_{\Lambda} M)$$

is an isomorphism of complexes.

Proof. We imitate [Pot13, Lem. 1.5(2)]. Consider the natural maps

$$\begin{aligned} N/I^n \otimes_{\Lambda} C^{\bullet}_{\mathrm{cts}}(G,M)/I^n C^{\bullet}_{\mathrm{cts}}(G,M) &\xrightarrow{\sim} N/I^n \otimes_{\Lambda} C^{\bullet}_{\mathrm{cts}}(G,M/I^n) \\ &\xrightarrow{\sim} C^{\bullet}_{\mathrm{cts}}(G,N/I^n \otimes_{\Lambda} M/I^n) \end{aligned}$$

where the first isomorphism holds due to the above lemma and the second isomorphism holds because all spaces are discrete and G^i is compact. The corollary follows by passing to the projective limit and noting that $N \otimes_{\Lambda} M \cong N \mathbin{\widehat{\otimes}_{\Lambda}} M$ as M is finitely generated. □

3.6.2 Perfectness of Group Cohomology

Lemma 3.6.12. *The complex $C^{\bullet}_{\mathrm{cts}}(G,T/I^n)$ consists of flat Λ/I^n-modules.*

Proof. The proof is a verbatim copy of the relevant part of the proof of [Pot13, Lem. 1.3(5)]. □

Lemma 3.6.13. *$C^{\bullet}_{\mathrm{cts}}(G,T)$ is idealwise separated for I.*

Proof. For a right ideal $\mathfrak{a}$ of Λ, which is finitely generated because Λ is right noetherian, we have

$$\mathfrak{a} \otimes_\Lambda C^\bullet_{\mathrm{cts}}(G,T) \xrightarrow{\sim} \varprojlim \left(\mathfrak{a} \otimes_\Lambda C^\bullet_{\mathrm{cts}}(G,T)\right) / \left(\mathfrak{a} \otimes_\Lambda I^n C^\bullet_{\mathrm{cts}}(G,T)\right)$$

by lemma 3.6.8. The claim now follows by noting that the images of $\mathfrak{a} \otimes_\Lambda I^n C^\bullet_{\mathrm{cts}}(G,T)$ and $\mathfrak{a} I^n \otimes_\Lambda C^\bullet_{\mathrm{cts}}(G,T)$ in $\mathfrak{a} \otimes_\Lambda C^\bullet_{\mathrm{cts}}(G,T)$ are the same, i.e. $\mathfrak{a} \otimes_\Lambda C^\bullet_{\mathrm{cts}}(G,T)$ is complete and in particular separated with respect to the filtration $[\mathfrak{a} I^n \otimes_\Lambda C^\bullet_{\mathrm{cts}}(G,T)]$, where $[-]$ denotes the image in $\mathfrak{a} \otimes_\Lambda C^\bullet_{\mathrm{cts}}(G,T)$. □

Proposition 3.6.14. *Assuming hypothesis 3.6.4, the complex $C^\bullet_{\mathrm{cts}}(G,T)$ consists of flat Λ-modules.*

Proof. Because I^n has the right Artin-Rees property and $C^\bullet_{\mathrm{cts}}(G,T)$ is idealwise separated, we can apply proposition 1.8.4. □

We are now able to deduce the following important structure statement for the group cohomology:

Theorem 3.6.15. *Assuming hypothesis 3.6.4,* $\mathbf{R}\Gamma_{\mathrm{cts}}(G,T) \in \mathbf{D}^{[0,e]}_{\mathrm{perf}}(\Lambda)$.

Proof. We imitate the proof of [Pot13, Cor. 1.2] with the necessary changes. The proof proceeds in several steps.

Step $\mathbf{R}\Gamma_{\mathrm{cts}}(G,T) \in \mathbf{D}^{[0,e]}(\Lambda)$: It suffices to show that $H^i_{\mathrm{cts}}(G,T)$ vanishes for $i > e$. Lemma 3.6.9 implies that the system $C^\bullet_{\mathrm{cts}}(G,T/I^n)$ has surjective transition maps, hence by a variant of [Wei94, Prop. 3.5.8] we find the exact sequence

$$0 \longrightarrow \mathbf{R}^1 \varprojlim H^{i-1}_{\mathrm{cts}}(G,T/I^n) \longrightarrow H^i_{\mathrm{cts}}(G,T) \longrightarrow \varprojlim H^i_{\mathrm{cts}}(G,T/I^n) \longrightarrow 0$$

for all i. Because T/I^n is a discrete and p-primary module, the p-primary module $H^i_{\mathrm{cts}}(G,T/I^n)$ vanishes for $i > e$ as G has p-cohomological dimension e, hence $H^i_{\mathrm{cts}}(G,T)$ vanishes for $i > e+1$. Regarding the vanishing of $H^{e+1}_{\mathrm{cts}}(G,T)$ we just have to check the left hand term in the exact sequence vanishes. Thus, it suffices to show that the system $H^e_{\mathrm{cts}}(G,T/I^n)$ has the Mittag-Leffler property. Considering the long exact sequence associated with the short exact sequence

$$0 \longrightarrow I^mT/I^nT \longrightarrow T/I^n \longrightarrow T/I^m \longrightarrow 0$$

we just have to check the vanishing of $H^{e+1}_{\mathrm{cts}}(G, I^mT/I^nT)$, which it indeed does.

Step $H^i_{\text{cts}}(N \otimes_\Lambda C^\bullet_{\text{cts}}(G,T)) = 0$ for $i \notin [0,e]$: For a finitely generated right Λ-module N the natural maps

$$N \otimes_\Lambda C^\bullet_{\text{cts}}(G,T) \longrightarrow N \widehat{\otimes}_\Lambda C^\bullet_{\text{cts}}(G,T) \longrightarrow C^\bullet_{\text{cts}}(G, N \otimes_\Lambda T)$$

are isomorphisms by lemma 3.6.8 and corollary 3.6.11. Hence, $H^i(N \otimes_\Lambda C^\bullet_{\text{cts}}(G,T)) = 0$ for $i \notin [0,e]$ follows from the last step and noting that this part did not require the considered module to be projective, just finitely generated.

Step $N \otimes^L_\Lambda \mathbf{R}\Gamma_{\text{cts}}(G,T) = [N \otimes_\Lambda C^\bullet_{\text{cts}}(G,T)]$: We would like to apply corollary 1.9.4, i.e. we have to check that $H^*(N \otimes_\Lambda C^\bullet_{\text{cts}}(G,T))$ vanishes from some point onwards for a finitely generated N, which is just the content of the last step.

Step $\mathbf{R}\Gamma_{\text{cts}}(G,T) \in \mathbf{D}^{[0,e]}_{\text{ft}}(\Lambda)$: We would like to apply proposition 1.9.2. Because $C^\bullet_{\text{cts}}(G,T)$ is I-adically complete, $C^\bullet_{\text{cts}}(G,T) \to \varprojlim C^\bullet_{\text{cts}}(G,T)/I^n$ is obviously a quasi-isomorphism (even an isomorphism), i.e. 1.9.2b) is true.

The proof of [Pot13, Lem. 1.3(4)] works verbatim in our more general situation, hence $H^i_{\text{cts}}(G, T/I)$ is finitely generated. Thus, $C^\bullet_{\text{cts}}(G, T/I)$ is quasi-isomorphic to a bounded above complex $D^\bullet$ of finitely generated, projective Λ/I-modules, see lemma 1.9.6. Because the complex $D^\bullet$ consists of flat modules and the last step we deduce the quasi-isomorphism in

$$\operatorname{gr}_I C^\bullet_{\text{cts}}(G,T) \cong \operatorname{gr}_I \Lambda \otimes_{\Lambda/I} C^\bullet_{\text{cts}}(G, T/I) \xleftarrow{\text{q.i.}} \operatorname{gr}_I \Lambda \otimes_{\Lambda/I} D^\bullet.$$

As $\operatorname{gr}_I \Lambda$ is left noetherian, $H^i(\operatorname{gr}_I \Lambda \otimes_{\Lambda/I} D^\bullet) \cong H^i(\operatorname{gr}_I C^\bullet_{\text{cts}}(G,T))$ is also finitely generated, i.e. 1.9.2a) holds.

Hence we can apply proposition 1.9.2 and deduce that $H^i_{\text{cts}}(G,T)$ is finitely generated for all i.

Step $\mathbf{R}\Gamma_{\text{cts}}(G,T) \in \mathbf{D}^{[0,e]}_{\text{perf}}(\Lambda)$: We would like to apply proposition 1.9.5 to deduce the statement. Since we already know that $\mathbf{R}\Gamma_{\text{cts}}(G,T) \in \mathbf{D}^{[0,e]}_{\text{ft}}(\Lambda)$ we just have to check that $\mathbf{R}\Gamma_{\text{cts}}(G,T)$ has tor amplitude $[0,e]$, i.e. $H^i(N \otimes^L_\Lambda \mathbf{R}\Gamma_{\text{cts}}(G,T))$ vanishes for $i \notin [0,e]$ and any finitely generated right R-module N. This however follows from step $H^i(N \otimes_\Lambda C^\bullet_{\text{cts}}(G,T)) = 0$ for $i \notin [0,e]$ and step $N \otimes^L_\Lambda \mathbf{R}\Gamma_{\text{cts}}(G,T) = [N \otimes_\Lambda C^\bullet_{\text{cts}}(G,T)]$. □

We record the following observation we made in the proof.

Corollary 3.6.16. *The derived tensor product $N \otimes^L_\Lambda \mathbf{R}\Gamma_{\text{cts}}(G,T)$ can be represented by $N \otimes_\Lambda C^\bullet_{\text{cts}}(G,T)$ in the derived category $\mathbf{D}(R)$.*

3.6.3 Base Change Properties

Hypothesis 3.6.17. Additionally to 3.6.4 assume the following:

(i) Λ' is an I'-adically complete $\mathcal{O}_K$-algebra as required in hypothesis 3.6.4(i); and

(ii) Y is a Λ'-Λ-bi-module which is finitely generated and projective as a Λ'-module and equipped with the induced topology by Λ' and the (right) Λ-action is continuous and commutes with the (left) Λ'-action, i.e. the ring homomorphism $\Lambda^{\mathrm{op}} \to \mathrm{End}_{\Lambda',\mathrm{cts}}(Y)$ is continuous.

Remark 3.6.18. For a (left) Λ-module M we denote the I'-adic completion of the (left) Λ'-module $Y \otimes_\Lambda M$ by

$$Y \mathbin{\widehat{\otimes}^{I'}_\Lambda} M = \varprojlim Y/(I')^n Y \otimes_\Lambda M.$$

This might differ from the completed tensor product. We note that the I'-adic completion also can be written as

$$\varprojlim Y/(I')^n Y \otimes_\Lambda M = \varprojlim Y/(I')^n Y \otimes_\Lambda M/I^{a(n)}M$$

because $\Lambda^{\mathrm{op}} \to \mathrm{End}_{\Lambda',\mathrm{cts}}(Y)$ is continuous and $(I')^n \mathrm{End}_{\Lambda',\mathrm{cts}}(Y)$ is closed in $\mathrm{End}_{\Lambda',\mathrm{cts}}(Y)$, hence there is an $a(n)$ such that $(I^{a(n)})^{\mathrm{op}}$ maps to $\mathrm{Hom}_{\Lambda',\mathrm{cts}}(Y, (I')^n Y)$.

Lemma 3.6.19. *The natural map*

$$Y \mathbin{\widehat{\otimes}^{I'}_\Lambda} C^\bullet_{\mathrm{cts}}(G, T) \to C^\bullet_{\mathrm{cts}}(G, Y \otimes_\Lambda T)$$

is an isomorphism of Λ'-complexes.

Proof. The proof is very similar to the proof of corollary 3.6.11. Consider the natural maps

$$\begin{aligned} Y/(I')^n Y \otimes_\Lambda C^\bullet_{\mathrm{cts}}(G,T)/I^{a(n)} &\xrightarrow{\sim} Y/(I')^n Y \otimes_\Lambda C^\bullet_{\mathrm{cts}}\left(G, T/I^{a(n)}T\right) \\ &\xrightarrow{\sim} C^\bullet_{\mathrm{cts}}\left(G, Y/(I')^n Y \otimes_\Lambda T/I^{a(n)}T\right) \end{aligned}$$

where the first isomorphism holds due to the lemma 3.6.9 and the second isomorphism holds because all spaces are discrete and G^i is compact. The lemma follows by passing to the projective limit and noting that $Y \otimes_\Lambda T \xrightarrow{\sim} Y \mathbin{\widehat{\otimes}^{I'}_\Lambda} T$ as T is finitely generated. $\square$

Theorem 3.6.20. *Assuming hypothesis 3.6.17, the natural map*

$$Y \otimes^L_\Lambda \mathbf{R}\Gamma_{\text{cts}}(G,T) \xrightarrow{\sim} \mathbf{R}\Gamma_{\text{cts}}(G, Y \otimes_\Lambda T)$$

is an isomorphism in $\mathbf{D}^{[0,e]}_{\text{perf}}(\Lambda')$.

Proof. Due to corollary 3.6.16, there is the equality

$$[Y \otimes_\Lambda C^\bullet_{\text{cts}}(G,T)] = Y \otimes^L_\Lambda \mathbf{R}\Gamma_{\text{cts}}(G,T),$$

i.e. it suffices to show that the natural map

$$Y \otimes_\Lambda C^\bullet_{\text{cts}}(G,T) \to C^\bullet_{\text{cts}}(G, Y \otimes_\Lambda T)$$

is a quasi-isomorphism. By the above lemma the problem is immediately reduced to showing that

$$Y \otimes_\Lambda C^\bullet_{\text{cts}}(G,T) \to Y \mathbin{\widehat{\otimes}}{}^{I'}_\Lambda C^\bullet_{\text{cts}}(G,T)$$

is a quasi-isomorphism.

We follow [Pot13, Lem. 1.5(3)]. Due to the perfectness of $\mathbf{R}\Gamma_{\text{cts}}(G,T)$ there is a quasi-isomorphism $P^\bullet \to C^\bullet_{\text{cts}}(G,T)$ such that $P^\bullet$ is a $[0,e]$-bounded complex of finitely generated, projective Λ-modules. From corollary 3.6.16 it follows that $Y \otimes_\Lambda P^\bullet \to Y \otimes_\Lambda C^\bullet_{\text{cts}}(G,T)$ and $Y/(I')^n \otimes_\Lambda P^\bullet \to Y/(I')^n \otimes_\Lambda C^\bullet_{\text{cts}}(G,T)$ are still quasi-isomorphisms. As $Y/(I')^n \otimes_\Lambda P^\bullet$ and $Y/(I')^n \otimes_\Lambda C^\bullet_{\text{cts}}(G,T)$ satisfy the Mittag-Leffler property, applying $\varprojlim$ yields another quasi-isomorphism $Y \mathbin{\widehat{\otimes}}{}^{I'}_\Lambda P^\bullet \to Y \mathbin{\widehat{\otimes}}{}^{I'}_\Lambda C^\bullet_{\text{cts}}(G,T)$.

However, note that $Y \otimes_\Lambda P^\bullet \to Y \mathbin{\widehat{\otimes}}{}^{I'}_\Lambda P^\bullet$ is an isomorphism as the P^i's are finitely generated. All in all, we get the following commutative diagram:

$$\begin{tikzcd}
Y \otimes_\Lambda P^\bullet \arrow[r, "\sim"] \arrow[d, "\wr"', "\text{q.i.}"] & Y \mathbin{\widehat{\otimes}}{}^{I'}_\Lambda P^\bullet \arrow[d, "\wr"', "\text{q.i.}"] \\
Y \otimes_\Lambda C^\bullet_{\text{cts}}(G,T) \arrow[r] & Y \mathbin{\widehat{\otimes}}{}^{I'}_\Lambda C^\bullet_{\text{cts}}(G,T)
\end{tikzcd}$$

which shows that the bottom arrow is an isomorphism in the derived category. □

3.6.4 Admissable Examples of Λ

We now want to identify classes of Λ which satisfy all the requirements in 3.6.4(i). Our first candidates are the Iwasawa algebras which play a central role in Iwasawa theory:

Proposition 3.6.21. *Let Λ be the Iwasawa algebra $\mathcal{O}_K[[G]]$ for a compact p-adic Lie group G, then the results of this section are valid for Λ, i.e. Λ fulfils hypothesis 3.6.4(i).*

Proof. Firstly, Λ and $\mathrm{gr}_I \Lambda$ are left and right noetherian by theorem 3.5.2. The Artin-Rees property follows from the next lemma. □

Lemma 3.6.22. *Assume that Λ is left and right noetherian and that Λ/I^n is finite for all n. Then the ideal I^n has the left and right Artin-Rees property.*

Proof. We follow [Neu88]. For the left Artin-Rees property we have to show that for every n and every left ideal $\mathfrak{a}$ there is a k such that $\mathfrak{a} \cap I^k \subseteq I^n\mathfrak{a}$.

As Λ is left noetherian, the left ideal $\mathfrak{a}$ is finitely generated, hence $\mathfrak{a}/I^n\mathfrak{a}$ is finitely generated over Λ/I^n, thus finite. We can deduce that the image of $f_k : \mathfrak{a} \cap I^k \to \mathfrak{a}/I^n\mathfrak{a}$ becomes stationary for $k \gg 0$. Fix such an k_0.

Consider the projection $f : \mathfrak{a} \to \mathfrak{a}/I^n\mathfrak{a}$, then

$$\bigcap_k f^{-1}(\mathrm{im}\, f_k) = \bigcap_k (\mathfrak{a} \cap I^k + I^n\mathfrak{a}) = \mathfrak{a} \cap \bigcap_k (I^k + I^n\mathfrak{a}) = \mathfrak{a} \cap \overline{I^n\mathfrak{a}}$$

where $\overline{I^n\mathfrak{a}}$ is the closure of $I^n\mathfrak{a}$ in Λ. As Λ is profinite, hence compact, every finitely generated left ideal of Λ is also compact, hence closed, i.e. $\overline{I^n\mathfrak{a}} = I^n\mathfrak{a}$ and the right hand side collapses to $I^n\mathfrak{a}$.

On the other hand we have that the intersection equals $f^{-1}(\mathrm{im}\, f_{k_0}) = \mathfrak{a} \cap I^{k_0} + I^n\mathfrak{a}$. Hence, we find $\mathfrak{a} \cap I^{k_0} \subseteq I^n\mathfrak{a}$ as desired. □

A second class of rings which feature heavily in arithmetic is the following:

Proposition 3.6.23. *Let Λ be a left and right noetherian π-adically complete $\mathcal{O}_K$-Banach algebra with $I = (\pi)$. Then the results of this section are valid for Λ, i.e. Λ fulfils hypothesis 3.6.4(i).*

Proof. The Rees rings $\Lambda[I^n x]$ are left and right noetherian because all I^n are generated by a central element (see [GW04, thm. 13.3]). Hence we can conclude using remark 3.6.6. □

Now we can deduce the following

Corollary 3.6.24. *Additionally to hypothesis 3.6.17 we assume that Λ fulfils the condition in proposition 3.6.21. Then, in the notation of the section 3.4, Λ_n fulfils hypothesis of proposition 3.6.23 for $n \geq 2$ and the natural map*

$$\Lambda_n \otimes^L_\Lambda \mathbf{R}\Gamma_{\mathrm{cts}}(G, T) \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{cts}}(G, \Lambda_n \otimes_\Lambda T)$$

is an isomorphism in $\mathbf{D}^{[0,e]}_{\mathrm{perf}}(\Lambda_n)$.

Proof. By theorem 3.5.2 we know that Λ_n is left and right noetherian for $n \geq 2$, i.e. Λ_n fulfils hypothesis of proposition 3.6.23, hence we can use theorem 3.6.20 in the following situation: $\Lambda' = Y = \Lambda_n$ and the Λ-action on Y is continuous because the $\langle I \rangle$-adic topology coincides with the π-adic topology (see the proof of theorem 3.4.4). □

Remark 3.6.25. There is also the obvious consequence for the group cohomology over K-Banach algebras, assuming that the situation is coming from an integral setting, as we have the isomorphism

$$\mathbf{R}\Gamma_{\mathrm{cts}}(G, T)[\pi^{-1}] \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{cts}}(G, T[\pi^{-1}])$$

in $\mathbf{D}(\Lambda[\pi^{-1}])$ because G is compact.

3.7 Continuous Group Cohomology over Fréchet-Stein Algebras

We now turn our attention to the cohomology over Fréchet-Stein algebras. Recall the definitions of §2.6 regarding the derived categories.

Definition 3.7.1. An $\mathcal{O}_K$-*model* or *integral model* of a K-algebra A is an $\mathcal{O}_K$-algebra Λ such that $K \otimes_{\mathcal{O}_K} -$ induces the original object. They are the obvious analogous definitions for integral modules of A-modules, etc.

From now on we will assume the following in order to get a well-behaved cohomology theory over Fréchet-Stein algebras.

Hypothesis 3.7.2. (i) $A_\infty = \varprojlim A_n$ is a two-sided Fréchet-Stein algebra and

(ii) $V_\infty = \varprojlim V_n$ is a locally projective, coadmissable module with a continuous $A_\infty[G]$-module structure, i.e. the group homomorphism $G \to \mathrm{End}_{A_\infty,\mathrm{cts}}(V_\infty)$ is continuous.

Remark 3.7.3. We note that the action of G on V_∞ also induces an action of G on V_n because we have the natural isomorphism

$$A_n \otimes_{A_\infty} \mathrm{End}_{A_\infty,\mathrm{cts}}(V_\infty) \cong \mathrm{End}_{A_n,\mathrm{cts}}(V_n).$$

Definition 3.7.4. We say hypothesis 3.7.2 *comes from a good integral situation* if the A_n-modules V_n have integral models T_n over $\mathcal{O}_K$-algebras Λ_n such that Λ_n with $I = (\pi)$ and T_n fulfil 3.6.4(i) or 3.6.4(iii) respectively. Furthermore, we require the transition maps to also have integral models.

Definition 3.7.5. Using the standard definition of the cohomology (see definition 3.6.3) we get the complex

$$\mathbf{R}\Gamma_{\mathrm{cts}}(G, V_\infty) \in \mathbf{D}(A_\infty).$$

However we also get

$$\mathbf{R}\Gamma_{\mathrm{sh,cts}}(G, V_\infty) \coloneqq \mathbf{R}\Gamma_{\mathrm{cts}}(G, V_n)_n \in \mathbf{D}_{\mathrm{sh}}(A_\infty)$$

as the image of the inverse system $C^\bullet_{\mathrm{cts}}(G, V_n)_n$ in the derived category $\mathbf{D}_{\mathrm{sh}}(A_\infty)$.

Regarding the second map of remark 2.6.5 we have:

Lemma 3.7.6. *The natural map*

$$\mathbf{R}\Gamma_{\mathrm{cts}}(G, V_\infty) \longrightarrow \mathbf{R}\varprojlim \mathbf{R}\Gamma_{\mathrm{sh,cts}}(G, V_\infty)$$

is an isomorphism in the derived category $\mathbf{D}(A_\infty)$.

Proof. As the maps $A_{n+1} \to A_n$ have dense image by theorem A (see theorem 2.4.7(i)), we have that the composite homomorphism

$$C^\bullet_{\mathrm{cts}}(G, V_{n+1}) \to A_n \otimes_{A_{n+1}} C^\bullet_{\mathrm{cts}}(G, V_{n+1}) \to A_n \widehat{\otimes}_{A_{n+1}} C^\bullet_{\mathrm{cts}}(G, V_{n+1}) \cong C^\bullet_{\mathrm{cts}}(G, V_n)$$

also has dense image. Hence, $C^\bullet_{\mathrm{cts}}(G, V_n)$ has the Mittag-Leffler property and we deduce

$$\mathbf{R}\varprojlim \mathbf{R}\Gamma_{\mathrm{cts}}(G, V_n) = [\varprojlim C^\bullet_{\mathrm{cts}}(G, V_n)],$$

noting that the natural isomorphism $C^\bullet_{\mathrm{cts}}(G, V_\infty) \xrightarrow{\sim} \varprojlim C^\bullet_{\mathrm{cts}}(G, V_n)$ is the universal property of the inverse limit. $\square$

Corollary 3.7.7. *Assuming that hypothesis 3.7.2 comes from a good integral situation the natural maps*

$$A_n \otimes_{A_{n+1}} H^*_{\mathrm{cts}}(G, V_{n+1}) \longrightarrow H^*_{\mathrm{cts}}(G, V_n) \qquad \text{and}$$
$$H^*_{\mathrm{cts}}(G, V_\infty) \longrightarrow \varprojlim H^*_{\mathrm{cts}}(G, V_n)$$

are isomorphisms and the cohomology groups $H^*_{\mathrm{cts}}(G, V_\infty)$ are coadmissable A_∞-modules.

Proof. As $A_{n+1} \to A_n$ is flat, the first isomorphism holds due to base change (see theorem 3.6.20). Hence, because $A_{n+1} \to A_n$ has dense image, the system $H^*_{\mathrm{cts}}(G, V_n)$ has the Mittag-Leffler property. Now, the second isomorphism follows from the above lemma.

As the groups $H^*_{\mathrm{cts}}(G, V_n)$ are finitely generated by the perfectness theorem 3.6.15, $H^*_{\mathrm{cts}}(G, V_\infty) \cong \varprojlim H^*_{\mathrm{cts}}(G, V_n)$ is a coadmissable module. □

Proposition 3.7.8. *Assuming that hypothesis 3.7.2 comes from a good integral situation, $\mathbf{R}\Gamma_{\mathrm{sh,cts}}(G, V_\infty)$ belongs to $\mathbf{D}^{[0,e]}_{\mathrm{sh,perf}}(A_\infty)$ and the natural map*

$$A_n \otimes^L_{A_\infty} \mathbf{R} \varprojlim \mathbf{R}\Gamma_{\mathrm{sh,cts}}(G, V_\infty) \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{cts}}(G, V_n)$$

is an isomorphism in $\mathbf{D}^{[0,e]}_{\mathrm{perf}}(A_n)$.

Proof. Local perfectness follows by observing that $\mathbf{R}\Gamma_{\mathrm{cts}}(G, V_n)$ is perfect over A_n due to the perfectness theorem 3.6.15.

Note that there is an isomorphism

$$\mathbf{R} \varprojlim \mathbf{R}\Gamma_{\mathrm{sh,cts}}(G, V_\infty) \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{cts}}(G, V_\infty)$$

by lemma 3.7.6. Furthermore, due to corollary 2.4.8(i), A_n is flat over A_∞. Hence, to check the isomorphism it suffices to show that the natural map

$$A_n \otimes_{A_\infty} H^*_{\mathrm{cts}}(G, V_\infty) \to H^*_{\mathrm{cts}}(G, V_n)$$

is an isomorphism, which is true since $H^*_{\mathrm{cts}}(G, V_\infty) \cong \varprojlim H^*_{\mathrm{cts}}(G, V_n)$ is coadmissable by the last lemma and by using corollary 2.4.8(ii). □

Lemma 3.7.9. *We assume that hypothesis 3.7.2 comes from a good integral situation. Furthermore, let $B_\infty = \varprojlim B_n$ be another two-sided Fréchet-Stein algebra and let $Y_\infty = \varprojlim Y_n$ be a B_∞-A_∞-bi-module such that Y_∞ is a coadmissable, projective right B_∞-module.*

Furthermore, we require the right action of A_∞ on Y_∞ to be continuous and to commute with the left action of B_∞, i.e. the ring homomorphism $A_\infty^{\mathrm{op}} \to \mathrm{End}_{B_\infty,\mathrm{cts}}(Y_\infty)$ is continuous. Additionally the induced action on Y_n via $B_n \otimes_{B_\infty} \mathrm{End}_{B_\infty,\mathrm{cts}}(Y_\infty) \to \mathrm{End}_{B_n,\mathrm{cts}}(Y_n)$ has to (continuously) factor through $A_{a(n)}^{\mathrm{op}}$ for some $a(n)$. Then $\varprojlim (Y_n \otimes_{A_\infty} V_\infty)$ is a locally projective, coadmissable B_∞-module.

If we moreover assume that B_∞ and Y_∞ come from a good integral situation as in definition 3.7.4 and also that the factorisation has an $\mathcal{O}_K$-model for every n then the natural map

$$\begin{aligned} Y_\infty \otimes^L_{A_\infty} \mathbf{R}\Gamma_{\mathrm{sh,cts}}(G, V_\infty) \coloneqq \left[Y_n \otimes^L_{A_{a(n)}} C^\bullet_{\mathrm{cts}}(G, V_{a(n)}) \right]_n \\ \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{sh,cts}}(G, (Y_n \otimes_{A_\infty} V_\infty)_n) \end{aligned}$$

is an isomorphism in $\mathbf{D}^{[0,e]}_{\mathrm{sh,perf}}(B_\infty)$.

Proof. As the actions of A_∞ and B_∞ on Y_∞ commute we find

$$B_n \otimes_{B_\infty} (Y_\infty \otimes_{A_\infty} V_\infty) \cong (B_n \otimes_{B_\infty} Y_\infty) \otimes_{A_\infty} V_\infty \cong Y_n \otimes_{A_\infty} V_\infty \cong Y_n \otimes_{A_{a(n)}} V_{a(n)}$$

where the last isomorphism holds because the action of A_∞ on Y_n factors through $A_{a(n)}$. The modules Y_n and $V_{a(n)}$ are finitely generated, thus $Y_n \otimes_{A_{a(n)}} V_{a(n)}$ is also finitely generated. Furthermore,

$$\begin{aligned} B_n \otimes_{B_n} (Y_{n+1} \otimes_{A_\infty} V_\infty) &\cong B_n \otimes_{B_{n+1}} B_{n+1} \otimes_{B_\infty} (Y_\infty \otimes_{A_\infty} V_\infty) \\ &\cong B_n \otimes_{B_\infty} (Y_\infty \otimes_{A_\infty} V_\infty) \\ &\cong Y_n \otimes_{A_\infty} V_\infty \end{aligned}$$

and we deduce that $\varprojlim (Y_n \otimes_{A_\infty} V_\infty)$ is a coadmissable B_∞-module.

The isomorphism in the derived category follows from theorem 3.6.20. □

Along the same lines we might also deduce a similar result on $\mathbf{R}\Gamma_{\mathrm{cts}}(G, V_\infty)$ which looks slightly less satisfying:

Corollary 3.7.10. *Under the same conditions as in the last lemma we find:*

$$\mathbf{R}\varprojlim \left(Y_n \otimes^L_{A_\infty} \mathbf{R}\Gamma_{\mathrm{cts}}(G, V_\infty) \right) \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{cts}}(G, \varprojlim (Y_n \otimes_{A_\infty} V_\infty)).$$

Proof. Applying $\mathbf{R}\varprojlim$ to lemma 3.7.9 together with lemma 3.7.6 gives the isomorphism

$$\mathbf{R}\varprojlim\left(Y_n \otimes^L_{A_{a(n)}} \mathbf{R}\Gamma_{\mathrm{cts}}(G, V_{a(n)})\right) \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{cts}}(G, \varprojlim(Y_n \otimes_{A_\infty} V_\infty)).$$

Hence to show the corollary it suffices to show the following isomorphism:

$$\begin{aligned} Y_n \otimes^L_{A_\infty} \mathbf{R}\Gamma_{\mathrm{cts}}(G, V_\infty) &\xrightarrow{\sim} Y_n \otimes^L_{A_{a(n)}} A_{a(n)} \otimes^L_{A_\infty} \mathbf{R}\Gamma_{\mathrm{cts}}(G, V_\infty) \\ &\xrightarrow{\sim} Y_n \otimes^L_{A_{a(n)}} \mathbf{R}\Gamma_{\mathrm{cts}}(G, V_{a(n)}), \end{aligned}$$

where the last isomorphism is by proposition 3.7.8. □

Remark 3.7.11. The isomorphism of the last corollary simplifies in some cases, for example if B_n stabilises, i.e. if $B_\infty = B_n$ for $n \geq n_0$. In this case we find:

$$Y_\infty \otimes^L_{A_\infty} \mathbf{R}\Gamma_{\mathrm{cts}}(G, V_\infty) \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{cts}}(G, Y_\infty \otimes_{A_\infty} V_\infty).$$

One important special case of the last lemma is the following:

Corollary 3.7.12. *In addition to the hypothesis of the last lemma, also assume that V_∞ is finitely presented and $\mathbf{R}\Gamma_{\mathrm{cts}}(G, V_\infty) \in \mathbf{D}^{[0,e]}_{\mathrm{coad.perf.}}(A_\infty)$. Then*

$$Y_\infty \otimes^L_{A_\infty} \mathbf{R}\Gamma_{\mathrm{cts}}(G, V_\infty) \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{cts}}(G, Y_\infty \otimes_{A_\infty} V_\infty)$$

is an isomorphism in $\mathbf{D}^{[0,e]}_{\mathrm{coad.perf.}}(B_\infty)$.

Proof. We note that finitely generated projective modules are finitely presented. Let M_∞ be a finitely presented A_∞-module. Then we check that $Y_\infty \otimes_{A_\infty} M_\infty$ is coadmissable, i.e. $Y_\infty \otimes_{A_\infty} M_\infty \cong \varprojlim(Y_n \otimes_{A_\infty} M_\infty)$. The statement follows from this observation as coadmissable modules are $\varprojlim$-acyclic due to theorem B (see theorem 2.4.7(ii)).

Assume that the finite presentation looks like

$$A_\infty^a \longrightarrow A_\infty^b \longrightarrow M_\infty \longrightarrow 0.$$

We would like to imitate the standard proof which shows that the projective limit of a surjective system commutes with the tensor product with a finitely presented module. Its only slightly non-trivial ingredient is that $\mathrm{im}(Y_n^a \to Y_n^b)$ has the Mittag-Leffler property assuming that $(Y_n^a)_n$ is a surjective system. Here we know that the transition maps of the system $(Y_n^a)_n$ are dense, hence $\mathrm{im}(Y_n^a \to Y_n^b)_n$ also has dense transition maps by looking at the ker-coker-sequence, i.e. the system $\mathrm{im}(Y_n^a \to Y_n^b)_n$ has the Mittag-Leffler property. □

Remark 3.7.13. In light of theorem 3.5.5 we see that V_∞ is always finitely presented in the most interesting situation.

Proposition 3.7.14. *Let Λ and T be as in corollary 3.6.24. Furthermore, let A_∞ be the associated Fréchet-Stein algebra to Λ as in §3.5. Then the natural map*

$$A_\infty \otimes^L_\Lambda \mathbf{R}\Gamma_{\mathrm{cts}}(G,T) \to \mathbf{R}\Gamma_{\mathrm{cts}}(G, A_\infty \otimes_\Lambda T)$$

is an isomorphism in $\mathbf{D}^{[0,e]}_{\mathrm{coad.perf.}}(A_\infty)$.

Proof. Due to the perfectness of $\mathbf{R}\Gamma_{\mathrm{cts}}(G,T)$ we find a perfect resolution $P^\bullet$. Note that it consists of projective modules which are not only finitely generated but also finitely presented as Λ is left noetherian, hence $A_\infty \otimes_\Lambda P^\bullet$ is a complex of coadmissable modules by 2.4.8(v). Then there are the following natural quasi-isomorphisms:

$$\begin{aligned}
A_\infty \otimes^L_\Lambda \mathbf{R}\Gamma_{\mathrm{cts}}(G,T) &\cong A_\infty \otimes^L_\Lambda [P^\bullet] && \\
&\cong [A_\infty \otimes_\Lambda P^\bullet] && P^\bullet \text{ projective} \\
&\cong \left[\left(\varprojlim A_n\right) \otimes_\Lambda P^\bullet\right] && \\
&\cong \left[\varprojlim \left(A_n \otimes_\Lambda P^\bullet\right)\right] && P^\bullet \text{ finitely presented} \\
&\cong \mathbf{R}\varprojlim\, [A_n \otimes_\Lambda P^\bullet] && A_\infty \otimes_\Lambda P^\bullet \text{ coadm., Theorem 2.4.7(ii)} \\
&\cong \mathbf{R}\varprojlim \left(A_n \otimes^L_\Lambda [P^\bullet]\right) && P^\bullet \text{ projective} \\
&\cong \mathbf{R}\varprojlim \left(A_n \otimes^L_\Lambda \mathbf{R}\Gamma_{\mathrm{cts}}(G,T)\right) && \\
&\cong \mathbf{R}\varprojlim \mathbf{R}\Gamma_{\mathrm{cts}}(G, A_n \otimes_\Lambda T) && \text{Cor. 3.6.24} \\
&\cong \mathbf{R}\Gamma_{\mathrm{cts}}(G, A_\infty \otimes_\Lambda T) && \text{Prop. 3.7.6.}
\end{aligned}$$

□

Remark 3.7.15. We note that in the situation of the last proposition the associated Fréchet-Stein algebra A_∞ and the coadmissable A_∞-module $A_\infty \otimes_\Lambda T$ canonically come from a good integral situation.

Furthermore, let Λ', I' and Y' be as in hypothesis 3.6.17. Set $Y_\infty := A'_\infty \otimes_{\Lambda'} Y$, which is a finitely presented, locally projective coadmissable left A'_∞-module with $Y_n := A'_n \otimes_{A'} Y$. We would like to define the induced right action of A_∞ on Y_∞.

Just like in remark 3.6.18 we find an $a(n)$ such that $(I^{a(n)})^{\mathrm{op}}$ maps to $(I')^n \operatorname{End}_{\Lambda'}(Y)$.

Hence there are induced continuous algebra maps

$$A_{a(n)}^{\mathrm{op}} \longrightarrow A'_n \otimes_{\Lambda'} \mathrm{End}_{\Lambda'}(Y) = \mathrm{End}_{A'_n}(Y_n)$$

compatible in n, independent of $a(n)$ and with an obvious integral model. The right action of A_∞ on $Y_\infty \cong \varprojlim Y_n$ now becomes obvious.

We conclude that the situation described here fulfils all the requirements of corollary 3.7.12, thus

$$Y_\infty \otimes^{L}_{A_\infty} \mathbf{R}\Gamma_{\mathrm{cts}}(G, A_\infty \otimes_A T) \xrightarrow{\sim} \mathbf{R}\Gamma_{\mathrm{cts}}(G, Y_\infty \otimes_A T).$$

4 Tate-Sen Theory with Non-Commutative Coefficients

In this chapter we extend the Tate-Sen theory developed by Berger-Colmez in [BC08, §3] to a setting with non-commutative coefficients. Since non-commutative aspects already appeared in their work while treating matrix rings we find that the same definitions still work and their proof methods remain valid.

4.1 The Tate-Sen Conditions

Let G_0 be a profinite group equipped with a continuous character $\chi : G_0 \to \mathbb{Z}_p^\times$ with open image, and let $H_0 := \ker \chi$. If $g \in G_0$, we denote the integer $\nu_p(\chi(g) - 1)$ with $n(g)$. Assuming that G is an open normal subgroup of G_0 we set $H = G \cap H_0 = G \cap \ker \chi$ which is also a normal subgroup of G_0. We define $\tilde{\Gamma}_H := G_0/H$ and we let C_H be its centre. G_0 acts via conjugation on the subgroup G/H of $\tilde{\Gamma}_H$ and there is another description of the action as $H = G \cap H_0$: G_0 acts on G/H via conjugation in G_0/H_0. The last group however is abelian, hence the action is trivial and $G/H \subset C_H$.

Lemma 4.1.1 ([BC08, Lem. 3.1.1])**.** *The group C_H is an open subgroup of $\tilde{\Gamma}_H$.*

We define $n_1(G)$ as the smallest integer $n \geq 1$ such that $\chi(G)$ contains $1 + p^n\mathbb{Z}_p$.

Hypothesis 4.1.2. Let S be a $\mathbb{Q}_p$-Banach algebra and let $\tilde{\Lambda}$ be an $\mathcal{O}_S$-ring equipped with a valuation $\nu_\Lambda : \tilde{\Lambda} \to \mathbb{R} \cup \{+\infty\}$ which fulfils:

(i) $\nu_\Lambda(x) = +\infty$ if and only if $x = 0$,

(ii) $\nu_\Lambda(xy) \geq \nu_\Lambda(x) + \nu_\Lambda(y)$,

(iii) $\nu_\Lambda(x + y) \geq \min(\nu_\Lambda(x), \nu_\Lambda(y))$, and

(iv) $\nu_\Lambda(p) > 0$ and $\nu_\Lambda(px) = \nu_\Lambda(p) + \nu_\Lambda(x)$ for $x \in \tilde{\Lambda}$.

The valuation can be used to define a separated topology on $\tilde{\Lambda}$. We assume that $\tilde{\Lambda}$ is complete for this topology.

Let $U \in M_d(\tilde{\Lambda})$, we define $\nu_\Lambda(U)$ as $\min(\nu_\Lambda(U_{ij}))$. Hence, we find

Lemma 4.1.3. *Let $U \in M_d(\tilde{\Lambda})$ with $\nu_\Lambda(U-1) > 0$. Then $U \in \mathrm{GL}_d(\tilde{\Lambda})$ and its inverse is $\sum_{n=0}^{+\infty}(1-U)^n$. Furthermore, $\nu_\Lambda(U) = \nu_\Lambda(U^{-1}) = 0$.*

Proof. We have to show that $V = \sum_{n=0}^{+\infty}(1-U)^n$ exists and that V is the inverse of U.

Regarding the existence, we have $\nu_\Lambda(U-1) > 0$ and $\nu_\Lambda(A \cdot B) \geq \nu_\Lambda(A) + \nu_\Lambda(B)$, hence $\nu_\Lambda((1-U)^n) \to \infty$ and the sum exists as all the entries of V converge.

Furthermore, V is indeed the inverse of U as

$$\begin{aligned} U \cdot V &= U \cdot \sum_{n=0}^{+\infty}(1-U)^n \\ &= (1-(1-U)) \cdot \sum_{n=0}^{c}(1-U)^n + U \cdot \sum_{n=c+1}^{+\infty}(1-U)^n \\ &= 1-(1-U)^{c+1} + U \cdot \sum_{n=c+1}^{+\infty}(1-U)^n. \end{aligned}$$

Now note that $(1-U)^{c+1}$ and $V_c = \sum_{n=c+1}^{+\infty}(1-U)^n$ go to zero as

$$(c+1)\nu_\Lambda(1-U) \leq \nu_\Lambda((1-U)^{c+1}) \leq \nu_\Lambda(V_c)$$

goes to infinity.

The same calculation shows that V is also a left-inverse. $\square$

We assume that $\tilde{\Lambda}$ is equipped with a continuous $\mathcal{O}_S$-linear action of G_0 such that $\nu_\Lambda(g(x)) = \nu_\Lambda(x)$ holds for all $g \in G_0$.

We define the Tate-Sen conditions:

Definition 4.1.4. In the following G denotes an open normal subgroup of G_0. The *Tate-Sen conditions* are as follows:

(TS1) There exists a $c_1 > 0$ such that for any two open subgroups $H_1 \subset H_2$ of H_0 which are normal in G there exists an $\alpha \in \tilde{\Lambda}^{H_1}$ such that $\nu_\Lambda(\alpha) > -c_1$ and $\sum_{\tau \in H_2/H_1} \tau(\alpha) = 1$.

(TS2) There is a $c_2 > 0$ and for every open subgroup H of H_0 which is normal in G there is an increasing sequence $(\Lambda_{H,n})_{n \geq n(H)}$ of closed $\mathcal{O}_S$-subrings of $\tilde{\Lambda}^H$. Furthermore, for $n \geq n(H)$ there is an $\mathcal{O}_S$-linear morphism $R_{H,n} : \tilde{\Lambda}^H \to \Lambda_{H,n}$. The previous data is subject to the following conditions:

a) If $H_1 \subset H_2$, then $\Lambda_{H_2,n} \subset \Lambda_{H_1,n}$ and the restriction of $R_{H_1,n}$ to $\tilde{\Lambda}^{H_2}$ coincides with $R_{H_2,n}$ (for $n \geq n(H_i)$, $i = 1, 2$);

b) $R_{H,n}$ is $\Lambda_{H,n}$-linear and $R_{H,n}(x) = x$ for $x \in \Lambda_{H,n}$;

c) $g(\Lambda_{H,n}) = \Lambda_{H,n}$ and $g(R_{H,n}(x)) = R_{H,n}(gx)$ for $g \in G_0$, in particular $R_{H,n}$ commutes with the action of $\tilde{\Gamma}_H$;

d) for $x \in \tilde{\Lambda}^H$, we have $\nu_\Lambda(R_{H,n}(x)) \geq \nu_\Lambda(x) - c_2$;

e) if $x \in \tilde{\Lambda}^H$, we have $R_{H,n}(x) \to x$ for $n \to +\infty$.

(TS3) There is a $c_3 > 0$ and there is for every open normal subgroup G of G_0 an integer $n(G) \geq \max(n(H), n_1(G))$, where $H = G \cap H_0$, such that for $\gamma \in \tilde{\Gamma}_H$ and $n \geq \max(n(G), n(\gamma))$ we have that $\gamma - 1$ is invertible on $X_{H,n} = \ker R_{H,n} = (1 - R_{H,n})(\tilde{\Lambda}^H)$ and $\nu_\Lambda((\gamma-1)^{-1}(x)) \geq \nu_\Lambda(x) - c_3$ if $x \in X_{H,n}$.

Remark 4.1.5. The morphisms $R_{H,n}$ are projection morphisms which give rise to the topological $\Lambda_{H,n}$-module decomposition $\tilde{\Lambda}^H = \Lambda_{H,n} \oplus X_{H,n}$.

Proposition 4.1.6. *Let $\tilde{\Lambda}$ a $\mathbb{Z}_p$-algebra which verifies the Tate-Sen conditions and S an orthonormalisable $\mathbb{Q}_p$-Banach algebra. Then $\mathcal{O}_S \widehat{\otimes}_{\mathbb{Z}_p} \tilde{\Lambda}$ with the tensor valuation also verifies the Tate-Sen conditions with the same constants.*

Proof. We first note that the tensor valuation

$$\nu_{\mathcal{O}_S \otimes_{\mathbb{Z}_p} \tilde{\Lambda}}(x) \coloneqq \sup \left\{ \min_i \nu_S(s_i)\nu(\lambda_i) \,\middle|\, x = \sum s_i \otimes \lambda_i \in \mathcal{O}_S \otimes_{\mathbb{Z}_p} \tilde{\Lambda} \right\}$$

fulfils all required properties in hypothesis 4.1.2 except possibly the first. However the completion is Hausdorff, hence $\tilde{\Lambda}' = \mathcal{O}_S \widehat{\otimes}_{\mathbb{Z}_p} \tilde{\Lambda}$ fulfils hypothesis 4.1.2.

An important building block of the proof is that $\mathcal{O}_S \widehat{\otimes} (\tilde{\Lambda})^H \to (\mathcal{O}_S \widehat{\otimes} \tilde{\Lambda})^H$ is an isomorphism. We follow [Bel15, Prop. D.1.3]. Because S is orthonormalisable there is an isometry between S and $c_I(\mathbb{Q}_p)$ (see definition 2.3.2). Thus, $\mathcal{O}_S$ is isomorphic to $c_I(\mathbb{Z}_p)$ and using this identification we find

$$(\mathcal{O}_S \widehat{\otimes}_{\mathbb{Z}_p} \tilde{\Lambda})^H \cong (c_I(\mathbb{Z}_p) \widehat{\otimes}_{\mathbb{Z}_p} \tilde{\Lambda})^H \overset{(1)}{\cong} c_I(\tilde{\Lambda})^H \overset{(2)}{\cong} c_I(\tilde{\Lambda}^H) \overset{(1)}{\cong} \mathcal{O}_S \widehat{\otimes}_{\mathbb{Z}_p} \tilde{\Lambda}^H$$

where (1) follows from the proof of lemma 2.3.5 and (2) holds as the Galois action on $c_I(\tilde{\Lambda})$ is given by $\sigma(c_i) = (\sigma c_i)$.

Regarding (TS1), we set $\alpha' \coloneqq 1 \otimes \alpha$, which fulfils, by construction of the valuation, $\nu_S(1 \otimes \alpha) \geq \nu(\alpha) > -c_1$.

Regarding (TS2), we set $\Lambda'_{H,n} \coloneqq \mathcal{O}_S \widehat{\otimes} \Lambda_{H,n}$ and $R'_{H,n} = \mathrm{id}_{\mathcal{O}_S} \widehat{\otimes} R_{H,n}$. We have to show that $\Lambda'_{H,n}$ are indeed closed subspaces of $\tilde{\Lambda}'$. That $\Lambda'_{H,n}$ maps injectively to $\tilde{\Lambda}'$ can be seen as above, which also implies that it is a (sequence) closed subset as a convergent sequence has only countably many terms.

Regarding (TS2d) and (TS3), we note that assuming that F is a $\mathbb{Z}_p$-linear function and that $\nu_\Lambda(F(x)) - \nu_\Lambda(x) \geq c$ for all $x \in \tilde{\Lambda}$ implies that $\nu_S(F_S(y)) - \nu_S(y) \geq c$ for all $y \in \tilde{\Lambda}'$ as it is true for $\mathcal{O}_S \otimes_{\mathbb{Z}_p} \tilde{\Lambda}$ because ν_S is the supremum of valuations taken over all possible representations and every representation occurring in the right valuation can be transferred to the left valuation. $\square$

Remark 4.1.7. The proof works in greater generality and just for simplicity we required S to be orthonormalisable. Since we are only interested in nc-affinoid algebras this is not a restriction.

4.2 Devissage for Continuous Cohomology

Lemma 4.2.1. *Let H be an open subgroup of H_0 and let $a > c_1$ and $k \in \mathbb{N}$. If $\tau \mapsto U_\tau$ is a continuous cocycle of H to $\mathrm{GL}_d(\tilde{\Lambda})$ which verifies $U_\tau - 1 \in p^k M_d(\tilde{\Lambda})$ and $\nu_\Lambda(U_\tau - 1) \geq a$ for $\tau \in H$, then there is a matrix $M \in \mathrm{GL}_d(\tilde{\Lambda})$ such that $M - 1 \in p^k M_d(\tilde{\Lambda})$ and $\nu_\Lambda(M-1) \geq a - c_1$. Furthermore, the cocycle $\tau \mapsto M^{-1} \cdot U_\tau \cdot \tau(M)$ satisfies $\nu_\Lambda(M^{-1} \cdot U_\tau \cdot \tau(M) - 1) \geq a + 1$.*

Proof. Let H_1 be a sufficiently[1] small open normal subgroup of H such that $\nu(U_\tau - 1) \geq a + 1 + c_1$ holds for $\tau \in H_1$. Let $\alpha \in \tilde{\Lambda}^{H_1}$ be the element given by TS1, which satisfies (A1) $\nu_\Lambda(\alpha) > -c_1$ and (A2) $\sum_{\tau \in H/H_1} \tau(\alpha) = 1$.

Define Q as a system of representatives of H/H_1 and set

$$M_Q = \sum_{\sigma \in Q} U_\sigma \cdot \sigma(\alpha).$$

Then (A2) implies $M_Q - 1 \in p^k M_d(\tilde{\Lambda})$ and $\nu_\Lambda(M_Q - 1) \geq a - c_1$, in particular, we find $\nu_\Lambda(M_Q - 1) > 0$, thus M_Q is invertible.

Let Q' be another system of representatives of H/H_1. Then the cocycle relation, the choice of H_1 and $\alpha \in \tilde{\Lambda}^{H_1}$ imply $\nu_\Lambda(M_Q - M_{Q'}) \geq a + 1$. Indeed, if $\sigma = \sigma'\tau'$, then

$$\begin{aligned} M_Q - M_{Q'} &= \sum_{\sigma \in Q} U_\sigma \sigma(\alpha) - \sum_{\sigma' \in Q'} U_{\sigma'} \sigma'(\alpha) \\ &= \sum_{\sigma' \in Q'} U_{\sigma'\tau'} \sigma'(\alpha) - \sum_{\sigma' \in Q'} U_{\sigma'} \sigma'(\alpha) \end{aligned}$$

[1] Use that $\nu(U - 1) > a + 1 + c_1$ is an open condition and the cocycle is continuous.

$$
\begin{aligned}
&= \sum_{\sigma' \in Q'} U_{\sigma'} \cdot \sigma'(U_{\tau'}) \cdot \sigma'(\alpha) - \sum_{\sigma' \in Q'} U_{\sigma'} \sigma'(\alpha) \\
&= \sum_{\sigma' \in Q'} U_{\sigma'} \cdot \sigma' \left(U_{\tau'} - 1\right) \cdot \sigma'(\alpha).
\end{aligned}
$$

Plugging in all the inequalities and noting that[2] $\nu_\Lambda(U_{\sigma'}) \geq 0$ yields the desired inequality.

At last, we find

$$
U_\tau \cdot \tau(M_Q) = \sum_{\sigma \in Q} U_\tau \cdot (\tau(U_\sigma) \cdot \tau\sigma(\alpha)) = \sum_{\sigma \in Q} U_{\tau\sigma} \cdot \tau\sigma(\alpha) = M_{\tau Q},
$$

hence

$$
M_Q^{-1} \cdot U_\tau \cdot \tau(M_Q) = M_Q^{-1} \cdot M_{\tau Q} = 1 + M_Q^{-1} \cdot (M_{\tau Q} - M_Q)
$$

and we deduce

$$
\nu_\Lambda(M_Q^{-1} \cdot U_\tau \cdot \tau(M_Q) - 1) \geq \nu_\Lambda(M_Q^{-1}) + \nu_\Lambda(M_{\tau Q} - M_Q) \geq 0 + a + 1
$$

as $\nu_\Lambda(M_Q^{-1}) = 0$ by construction. □

Corollary 4.2.2. *Let H be an open subgroup of H_0 and let $a > c_1$ and $k \in \mathbb{N}$. If $\tau \mapsto U_\tau$ is a continuous cocycle of H to $\mathrm{GL}_d(\tilde{\Lambda})$ which verifies $U_\tau - 1 \in p^k M_d(\tilde{\Lambda})$ and $\nu_\Lambda(U_\tau - 1) \geq a$ for $\tau \in H$, then there exists a matrix $M \in \mathrm{GL}_d(\tilde{\Lambda})$ such that $M - 1 \in p^k M_d(\tilde{\Lambda})$, $\nu_\Lambda(M-1) \geq a - c_1$ and the cocycle $\tau \mapsto M^{-1} \cdot U_\tau \cdot \tau(M)$ is trivial.*

Proof. Applying the previous lemma inductively, one constructs matrices $(M_m)_m$ such that $M_m - 1 \in p^k M_d(\tilde{\Lambda})$ and $\nu_\Lambda(M_m - 1) \geq a - c_1 + m - 1$. Furthermore, the cocycle $\tau \mapsto U_{n,\tau} := (\prod_{m=1}^n M_m)^{-1} \cdot U_\tau \cdot \tau(\prod_{m=1}^n M_m)$ satisfies $\nu_\Lambda(U_{n,\tau} - 1) \geq a + n$ for $\tau \in H$. The product $\prod_{m=1}^\infty M_m$ converges to an $M \in \mathrm{GL}_d(\tilde{\Lambda})$ which has all the desired properties. □

Lemma 4.2.3. *Let G be an open normal subgroup of G_0 and set $H = G \cap H_0$. Pick $\gamma \in \tilde{\Gamma}_H$ and n such that $n \geq \max(n(\gamma), n(G))$. Let $\delta > 0$, $a, b \in \mathbb{R}$ satisfying $a \geq c_2 + c_3 + \delta$ and $b \geq \max(a + c_2, 2c_2 + 2c_3 + \delta)$. Assume $U = 1 + p^k U_1 + p^k U_2$ with*

$$
\begin{aligned}
U_1 &\in M_d(\Lambda_{H,n}), \qquad \nu_\Lambda(U_1) \geq a - \nu_\Lambda(p^k) \qquad \textit{and} \\
U_2 &\in M_d(\tilde{\Lambda}^H), \qquad \nu_\Lambda(U_2) \geq b - \nu_\Lambda(p^k).
\end{aligned}
$$

[2]Follows by $\nu_\Lambda(U - 1) \geq 0$, hence, using the strong triangle inequality, we find $\nu_\Lambda(U) \geq \nu_\Lambda(1) = 0$, where the last equality is implied by the fourth condition on the valuation.

Then there exists an $M \in 1 + p^k M_d(\tilde{\Lambda}^H)$ *satisfying* $\nu_\Lambda(M-1) \geq b - c_2 - c_3$ *such that* $M^{-1} \cdot U \cdot \gamma(M) = 1 + p^k V_1 + p^k V_2$ *with*

$$\begin{aligned} V_1 \in M_d(\Lambda_{H,n}), \qquad & \nu_\Lambda(V_1) \geq a - \nu_\Lambda(p^k) \qquad \textit{and} \\ V_2 \in M_d(\tilde{\Lambda}^H), \qquad & \nu_\Lambda(V_2) \geq b - \nu_\Lambda(p^k) + \delta. \end{aligned}$$

Proof. By (TS2) and (TS3), we can write U_2 as

$$U_2 = R_{H,n}(U_2) + (1-\gamma)(V)$$

with $\nu_\Lambda(R_{H,n}(p^k U_2)) \geq b - c_2 \geq a$ (TS2d) and $\nu_\Lambda(p^k V) \geq b - c_2 - c_3 > 0$ (TS3).

Set

$$\begin{aligned} M &= 1 + p^k V, \\ V_1 &= U_1 + R_{H,n}(U_2) \in M_d(\Lambda_{H,n}) \qquad \text{and} \\ p^k V_2 &= M^{-1} \cdot U \cdot \gamma(M) - (1 + p^k V_1). \end{aligned}$$

The only open claim is $\nu_\Lambda(p^k V_2) \geq b + \delta$. Note that

$$p^k V_2 = (1 - p^k V + \cdots) \cdot (1 + p^k U_1 + p^k U_2) \cdot (1 + p^k \gamma(V)) - (1 + p^k U_1 + p^k R_{H,n}(U_2)).$$

It is beneficial to write $p^k V_2$ formally as $W_0 + p^k W_1 + p^{2k} W_2 + \cdots$ and it is clear that W_0 vanishes. Regarding W_1, we have

$$W_1 = -V + U_1 + U_2 + \gamma(V) - (U_1 + R_{H,n}(U_2)) = U_2 - (1-\gamma)(V) - R_{H,n}(U_2) = 0.$$

Moreover, every summand of $p^{nk} W_n$ for $n \geq 2$ has the following form by construction: it is a product of at most one factor of either $p^k U_1$ or $p^k U_2$, at most one factor of $p^k \gamma(V)$ and all other factors are of the form $p^k V$. Hence, considering that $\nu_\Lambda(p^k V) = \nu_\Lambda(p^k \gamma(V))$ we find

$$\begin{aligned} \nu_\Lambda(p^{nk} W_k) &\geq (n-1) \cdot \nu_\Lambda(p^k V) + \min(\nu_\Lambda(p^k V), \nu_\Lambda(p^k U_1), \nu_\Lambda(p^k U_2)) \\ &\geq (b - c_2 - c_3) + \min(b - c_2 - c_3, a, b) \\ &\geq b + \min(b - 2c_2 - 2c_3, a - c_2 - c_3, b - c_2 - c_3) \\ &\geq b + \delta. \end{aligned}$$

We deduce $\nu_\Lambda(p^k V_2) \geq b + \delta$. $\square$

Corollary 4.2.4. *Let G be an open normal subgroup of G_0 and set $H = G \cap H_0$. Pick $\gamma \in \tilde{\Gamma}_H$ and n such that $n \geq \max(n(\gamma), n(G))$. Let $\delta > 0$ and let $b \geq 2c_2 + 2c_3 + \delta$. Assume $U \in 1 + p^k M_d(\tilde{\Lambda}^H)$ verifies $\nu_\Lambda(U - 1) \geq b$. Then there exists an $M \in 1 + p^k M_d(\tilde{\Lambda}^H)$ such that $\nu_\Lambda(M - 1) \geq b - c_2 - c_3$ and*

$$M^{-1} \cdot U \cdot \gamma(M) \in 1 + p^k M_d(\Lambda_{H,n}).$$

Proof. Set $a_n = b - c_2$ and $b_n = b + n\delta$. Applying the previous lemma inductively (with the constants a_n, b_n) yields M_m with $M_m \in 1 + p^k M_d(\tilde{\Lambda}^H)$ and $\nu_\Lambda(M_m - 1) \geq b_{m-1} - c_2 - c_3$ such that $U_n = (\prod_{m=1}^n M_m)^{-1} \cdot U \cdot \gamma(\prod_{m=1}^n M_m)$ can be decomposed as $U_n = 1 + p^k U_{n,1} + p^k U_{n,2}$ with:

$$\begin{aligned} U_{n,1} &\in M_d(\Lambda_{H,n}), & \nu_\Lambda(U_{n,1}) &\geq a_n - \nu_\Lambda(p^k), \\ U_{n,2} &\in M_d(\tilde{\Lambda}^H), & \nu_\Lambda(U_{n,2}) &\geq b_n - \nu_\Lambda(p^k). \end{aligned}$$

Here we have to start the inductive definition with $U_{0,1} = 0$ and $p^k U_{0,2} = U - 1$. At last we see that

$$1 + \lim_{n\to\infty} U_{n,1} = \lim_{n\to\infty} U_n = M^{-1} \cdot U \cdot \gamma(M),$$

with $M \coloneqq \prod_{m=1}^\infty M_m$, indeed converges to an element in $M_d(\Lambda_{H,n})$ because $\Lambda_{H,n}$ is closed in $\tilde{\Lambda}^H$. $\square$

Lemma 4.2.5. *Let G be an open normal subgroup of G_0 and set $H = G \cap H_0$. Pick $\gamma \in \tilde{\Gamma}_H$ and n such that $n \geq \max(n(\gamma), n(G))$. Let $B \in \mathrm{GL}_d(\tilde{\Lambda}^H)$. If there are $V_i \in \mathrm{GL}_d(\Lambda_{H,n})$ for $i \in \{1, 2\}$ such that $\nu_\Lambda(V_i - 1) > c_3$ and $\gamma(B) = V_1 \cdot B \cdot V_2$, then $B \in \mathrm{GL}_d(\Lambda_{H,n})$.*

Proof. Let $C = B - R_{H,n}(B)$, then $\gamma(C) = V_1 \cdot C \cdot V_2$ as $R_{H,n}$ is $\Lambda_{H,n}$-linear and commutes with γ. Moreover,

$$\begin{aligned} \gamma(C) - C &= V_1 \cdot C \cdot V_2 - C \\ &= (V_1 - 1) \cdot C \cdot V_2 + V_1 \cdot C \cdot (V_2 - 1) - (V_1 - 1) \cdot C \cdot (V_2 - 1). \end{aligned}$$

Hence, if $\nu_\Lambda(C)$ is finite we have

$$\nu_\Lambda((\gamma - 1)(C)) \geq \nu_\Lambda(C) + \min\left(\nu_\Lambda(V_1 - C), \nu_\Lambda(V_2 - 1)\right) > \nu_\Lambda(C) + c_3$$

which contradicts (TS3). Hence $C = 0$. Applying the same reasoning to B^{-1} yields the desired conclusion. $\square$

Proposition 4.2.6. *Let $\sigma \mapsto U_\sigma$ be a continuous cycle of G_0 with values in $\mathrm{GL}_d(\tilde{\Lambda})$. Assume that there is an open normal subgroup G of G_0 such that $U_\sigma - 1 \in p^k M_d(\tilde{\Lambda})$ and $\nu_\Lambda(U_\sigma - 1) > c_1 + 2c_2 + 2c_3$ for $\sigma \in G$. Set $H = G \cap H_0$. Then there exists an $M \in 1 + p^k M_d(\tilde{\Lambda})$ with $\nu_\Lambda(M-1) > c_2 + c_3$ such that the cocycle $\sigma \mapsto V_\sigma = M^{-1} \cdot U_\sigma \cdot \sigma(M)$ is trivial on H and takes values in $\mathrm{GL}_d(\Lambda_{H,n(G)})$.*

Proof. Corollary 4.2.2 (with $a = c_1 + 2c_2 + 2c_3$) provides an $M_1 \in 1 + p^k M_d(\tilde{\Lambda})$ with $\nu_\Lambda(M_1 - 1) > 2c_2 + 2c_3$ such that the restriction of the cocycle $\sigma \mapsto U'_\sigma = M_1^{-1} \cdot U_\sigma \cdot \sigma(M_1)$ to H is trivial, hence the cocycle is an inflation of a cocycle $\tilde{\Gamma}_H = G_0/H \to \mathrm{GL}_d(\tilde{\Lambda}^H)$.

Choose a $\gamma \in G/H \subset C_H$ with $n(\gamma) = n(G)$, hence $U'_\gamma - 1 \in p^k M_d(\tilde{\Lambda}^H)$ and $\nu_\Lambda(U'_\gamma - 1) > 2c_2 + 2c_3$. In this case corollary 4.2.4 yields a matrix $M_2 \in 1 + p^k M_d(\tilde{\Lambda}^H)$ with $\nu_\Lambda(M_2 - 1) > c_2 + c_3$ such that $M_2^{-1} \cdot U'_\gamma \cdot \gamma(M_2) \in \mathrm{GL}_d(\Lambda_{H,n(G)})$.

Let $M = M_1 \cdot M_2$. Then $M \in 1 + p^k M_d(\tilde{\Lambda})$, $\nu_\Lambda(M - 1) > c_2 + c_3$ and the cocycle $\tau \mapsto V_\tau = M^{-1} U_\tau \tau(M)$ is trivial on H, has values in $\mathrm{GL}_d(\tilde{\Lambda}^H)$, verifies $V_\gamma \in \mathrm{GL}_d(\Lambda_{H,n(G)})$ and $\nu_\Lambda(V_\gamma - 1) > c_2 + c_3 > c_3$.

If $\tau \in G_0$, then $\tau\gamma = \gamma\tau$ holds in $\tilde{\Gamma}_H = G_0/H$ (see the discussion at the beginning of section 4.1) and the cocycle relation yields

$$V_\tau \tau(V_\gamma) = V_{\tau\gamma} = V_{\gamma\tau} = V_\gamma \gamma(V_\tau),$$

i.e. $\gamma(V_\tau) = V_\gamma^{-1} V_\tau \tau(V_\gamma)$. Hence, lemma 4.2.5 implies that the cocycle $\tau \mapsto V_\tau$ actually takes values in $\mathrm{GL}_d(\Lambda_{H,n(G)})$. □

Corollary 4.2.7. *Let $\sigma \mapsto U_\sigma$ and $\sigma \mapsto U'_\sigma$ be two continuous cocycles of G_0 with values in $\mathrm{GL}_d(\tilde{\Lambda})$ and $\mathrm{GL}_{d'}(\tilde{\Lambda})$ respectively, fulfilling the requirements of the proposition. Additionally we assume that all α of (TS1) are in the centre of $\tilde{\Lambda}$. Furthermore we assume that there is an $\mathcal{O}_S$-matrix A such that*

$$A \cdot U_\sigma = U'_\sigma \cdot A$$

holds for all $\sigma \in G_0$. Then

$$A \cdot X = X' \cdot A$$

holds for $X \in \{M, M^{-1}, V_\sigma\}$ in the proposition.

Proof. In order to lighten the notation we say that X is A-equivariant if $A \cdot X = X' \cdot A$ holds.

We can check separately the A-equivariance of M_1 and M_2, in this case the A-equivariance of V_σ follows directly.

Regarding lemma 4.2.1, we see that M is defined by

$$M = \sum_{\sigma \in Q} U_\sigma \cdot \sigma(\alpha).$$

Hence, from $A \cdot U_\sigma = U'_\sigma \cdot A$ it follows that $A \cdot M = M' \cdot A$ because α is central. Since M is invertible we also find $(M')^{-1} \cdot A = A \cdot M^{-1}$, hence also the cocycle $M^{-1} \cdot U_\tau \cdot \tau(M)$ is A-equivariant. It follows by induction that M_1 produced by corollary 4.2.2 as well as the new cocycle are A-equivariant.

Moreover regarding lemma 4.2.3, $R_{H,n}$ is $\mathcal{O}_S$-linear, hence

$$U'_2 \cdot A = R_{H,n}(U'_2 \cdot A) + (1-\gamma)(A \cdot V).$$

Since we are in the range where $1-\gamma$ is bijective we deduce that $A \cdot V = V' \cdot A$, thus M, V_1 and V_2 are also indeed A-equivariant. Again, by induction M_2 is A-equivariant. □

Remark 4.2.8. If $\tilde{\Lambda}$ as in proposition 4.1.6 is commutative, then $\mathcal{O}_S \,\widehat{\otimes}_{\mathbb{Z}_p} \tilde{\Lambda}$ has the property that the α of (TS1) are indeed in the centre by construction.

4.3 The Method of Tate-Sen and S-representations

We denote with $\tilde{\Lambda}^+$ (resp. $\Lambda^+_{H,n}$) the ring of integers of $\tilde{\Lambda}$ (resp. $\Lambda_{H,n}$), i.e. the elements with non-negative valuation.

To ease the notation we denote with $M_d^{\mathrm{op}}(R)$ the ring $M_d(R^{\mathrm{op}})$, and we write $\star$ for the product in this ring to distinguish it from the usual matrix product $\cdot$, i.e. $A \star B = (B^t \cdot A^t)^t$. (Similarly for $\mathrm{GL}_d^{\mathrm{op}}(R)$.)

Proposition 4.3.1. *Let T be a finitely generated, projective left $\mathcal{O}_S$-module equipped with a continuous $\mathcal{O}_S$-linear action of G_0 and let k be an integer with $\nu_\Lambda(p^k) > c_1 + 2c_2 + 2c_3$. Assume that all α of (TS1) are in the centre of $\tilde{\Lambda}$ and that G is an open normal subgroup of G_0 which acts trivially on T/p^kT. Set $H = G \cap H_0$ and choose $n \geq n(G)$. Then $\tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} T$ equipped with the diagonal G_0-action contains a functorial projective $\Lambda^+_{H,n}$-submodule $\mathbf{D}^+_{H,n}(T)$ with the following properties:*

(i) $\mathbf{D}^+_{H,n}(T)$ is point-wise fixed by H and stable under G_0,

(ii) the natural homomorphism $\tilde{\Lambda}^+ \otimes_{\Lambda^+_{H,n}} \mathbf{D}^+_{H,n}(T) \to \tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} T$ is an isomorphism, and

(iii) $\mathbf{D}^+_{H,n}(T)$ *possesses a projective* $\Lambda^+_{H,n}$*-basis which is* c_3*-fixed by* G/H*, i.e. for every* $\gamma \in G/H$ *let* W_γ *be the matrix of* γ *with respect to the basis, then* $\nu_\Lambda(W_\gamma - 1) > c_3$*.*

Assuming that T *is free of rank* d*, then* $\mathbf{D}^+_{H,n}(T)$ *is also free of rank* d *and it fulfils the stronger version (iii)' of (iii):*

(iii)' the projective basis in part (iii) can be taken to be a basis.

Moreover $\mathbf{D}^+_{H,n}(T)$ *is the unique* $\Lambda^+_{H,n}$*-submodule with properties (i), (ii) and (iii)'.*

Proof. Since T is projective there exists finitely generated $\mathcal{O}_S$-modules P and F such that $F = T \oplus P$ is free, say of rank d. If T is free we set $F = T$. We can extend the G_0-action of T to F by setting $\sigma(t,p) := (\sigma(t), p)$ for $\sigma \in G_0$. Obviously G acts trivially on F/p^kF.

Let (v_i) be a left $\mathcal{O}_S$-basis of F and let $U_\sigma = (u^\sigma_{i,j})$ be the matrix of σ in $\mathrm{GL}^{\mathrm{op}}_d(\mathcal{O}_S)$. Note that $G_0 \to \mathrm{GL}^{\mathrm{op}}_d(\mathcal{O}_S)$ is continuous and can be regarded as a cocycle with values in $\mathrm{GL}^{\mathrm{op}}_d(\mathcal{O}_S) \subset \mathrm{GL}^{\mathrm{op}}_d(\tilde{\Lambda}^+)$. We assumed that $U_\sigma \in 1 + p^k M^{\mathrm{op}}_d(\mathcal{O}_S)$ for $\sigma \in G$ and proposition 4.2.6 provides a matrix $M \in \mathrm{GL}^{\mathrm{op}}_d(\tilde{\Lambda})$ with $\nu_\Lambda(M-1) > c_2 + c_3$ (hence $M \in \mathrm{GL}^{\mathrm{op}}_d(\tilde{\Lambda}^+)$) such that the cocycle $\sigma \mapsto V_\sigma = M^{-1} \star U_\sigma \star \sigma(M)$ is trivial on H and takes values in $\mathrm{GL}^{\mathrm{op}}_d(\Lambda_{H,n(G)}) \cap \mathrm{GL}^{\mathrm{op}}_d(\tilde{\Lambda}^+) = \mathrm{GL}^{\mathrm{op}}_d(\Lambda^+_{H,n(G)})$.

Let f_F be the automorphism of $\tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} F$ associated with M, then set $e_k = f_F(v_j) = \sum_j m_{j,k} v_j$ if $M = (m_{j,k})$. Furthermore, let $\mathbf{D}^+_{H,n}(F)$ be the left $\Lambda^+_{H,n}$-submodule of $\tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} F$ which is generated by (e_k), i.e. $\mathbf{D}^+_{H,n}(F) = f_F(\Lambda^+_{H,n} \otimes_{\mathcal{O}_S} F)$. It is obviously free and of rank d.

The formation of $\mathbf{D}^+_{H,n}(F)$ is functorial as can be seen as follows: let $f : F \to F'$ be a G_0-equivariant morphism of free left $\mathcal{O}_S$-modules which fulfil all the requirements. We denote the objects associated with F and F' without a prime or with a prime respectively and by abuse of notation we also write f for $\tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} f$. Let A be the matrix of the map f in the basis chosen above. Then $A \star M = M' \star A$ can be deduced from corollary 4.2.7, i.e. $f \circ f_F = f_{F'} \circ f$ holds. Then

$$\begin{aligned} f(\mathbf{D}^+_{H,n}(F)) = f \circ f_F(\Lambda^+_{H,n} \otimes_{\mathcal{O}_S} F) &= f_{F'} \circ f(\Lambda^+_{H,n} \otimes_{\mathcal{O}_S} F) \\ &\subseteq f_{F'}(\Lambda^+_{H,n} \otimes_{\mathcal{O}_S} F') = \mathbf{D}^+_{H,n}(F'), \end{aligned}$$

i.e. $\tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} f$ maps $\mathbf{D}^+_{H,n}(F) \subset \tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} F$ to $\mathbf{D}^+_{H,n}(F') \subset \tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} F'$. We call this G_0-equivariant morphism $\mathbf{D}^+_{H,n}(f)$.

We now show properties (i)-(iii) for $\mathbf{D}^+_{H,n}(F)$.

Regarding (i), note that

$$\begin{aligned}
\sigma(e_k) &= \sum_j \sigma(m_{j,k}) \cdot \sigma(v_j) \\
&= \sum_j \sigma(m_{j,k}) \cdot \left(\sum_i u_{i,j}^{\sigma} \cdot v_i\right) \\
&= \sum_i \left(\sum_j \sigma(m_{j,k}) \cdot u_{i,j}^{\sigma}\right) v_i \\
&= \sum_i (U_\sigma \star \sigma(M))_{i,k} \cdot v_i \\
&= \sum_i (M \star V_\sigma)_{i,k} \cdot v_i \\
&= \sum_i \left(\sum_j v_{j,k}^{\sigma} \cdot m_{i,j}\right) \cdot v_i \\
&= \sum_j v_{j,k}^{\sigma} \cdot \left(\sum_i m_{i,j} \cdot v_i\right) \\
&= \sum_j v_{j,k}^{\sigma} \cdot e_j,
\end{aligned}$$

hence the submodule $\mathbf{D}_{H,n}^{+}(F)$ is G_0 stable and its basis is fixed by H.

(ii) is trivially fulfilled as (e_k) is by construction also a $\tilde{\Lambda}^{+}$-basis of $\tilde{\Lambda}^{+} \otimes_{\mathcal{O}_S} F$.

Regarding (iii)', let W_γ be the matrix of $\gamma \in G/H$ associated with the basis (e_k), then $W_\gamma = M^{-1} \star U_\sigma \star \sigma(M)$ for some lift σ of γ. Hence:[3]

$$\begin{aligned}
\nu_\Lambda(W_\gamma - 1) &= \nu_\Lambda(M^{-1} \star U_\sigma \star \sigma(M) - 1) \\
&\geq \min\left(\nu_\Lambda(M^{-1} - 1), \nu_\Lambda(U_\sigma - 1), \nu_\Lambda(\sigma(M) - 1)\right) \\
&> \min\left(c_2 + c_3, c_1 + 2c_2 + 2c_3, c_2 + c_3\right) \\
&> c_3.
\end{aligned}$$

We now show that there can only be one module with properties (i)-(iii)' if $T = F$ is free. Choose any $\gamma \in C_H$ with $n(\gamma) = n$. Let (e_k^i) for $i = 1, 2$ be two bases as in (iii)' which are fixed by H, such that the matrices $W_\gamma^i \in \mathrm{GL}_d^{\mathrm{op}}(\Lambda_{H,n(G)}^{+})$ satisfy $\nu_\Lambda(W_\gamma^i - 1) > c_3$. Let $B \in \mathrm{GL}_d^{\mathrm{op}}(\tilde{\Lambda}^{+})$ be the matrix of (e_k^2) in the basis (e_k^1). Hence, B is invariant under H and we have $W_\gamma^1 = B^{-1} \cdot W_\gamma^2 \cdot \gamma(B)$. By lemma 4.2.5, this implies that $B \in \mathrm{GL}_d^{\mathrm{op}}(\Lambda_{H,n})$, hence $B \in \mathrm{GL}_d^{\mathrm{op}}(\Lambda_{H,n}^{+})$. Thus, both $\Lambda_{H,n}^{+}$-modules generated by (e_k^i) for $i = 1, 2$ coincide.

[3] Using $-1 + \prod x_i = -1 + \prod(1 + (x_i - 1)) = \sum(x_i - 1) + $ terms of higher order.

Let $\iota_X : X \to F$ and $p_X : F \to X$ be the inclusion and the projection maps of $X \in \{T, P\}$ associated with the direct sum decomposition $F = T \oplus P$. By abuse of notation we also denote the maps $\tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} \iota_X$ and $\tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} p_X$ by ι_X and p_X respectively again. Note that $\iota_X : \tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} X \to \tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} F$ is injective and we identify $\tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} X$ with its image in $\tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} F$.

Define

$$\mathbf{D}^+_{H,n}(T) \coloneqq p_T \circ f_F \circ \iota_T(\Lambda^+_{H,n} \otimes_{\mathcal{O}_S} T) \subset \tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} T.$$

Let π_T be the G_0-equivariant endomorphism $\iota_T \circ p_T$ of $\tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} F$. Then as already shown above $\pi_T \circ f_F = f_F \circ \pi_T$ holds. Since ι_T is injective we deduce

$$\mathbf{D}^+_{H,n}(T) \cong \pi_T \circ f_F \circ \iota_T(\Lambda^+_{H,n} \otimes_{\mathcal{O}_S} T) = f_F \circ \iota_T(\Lambda^+_{H,n} \otimes_{\mathcal{O}_S} T) = f_F(\Lambda^+_{H,n} \otimes_{\mathcal{O}_S} T)$$

which implies $f_F(\Lambda^+_{H,n} \otimes_{\mathcal{O}_S} T) \subset (\tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} T) \cap \mathbf{D}^+_{H,n}(F)$. Moreover we find

$$\mathbf{D}^+_{H,n}(T) = p_T \circ f_F \circ \pi_T(\Lambda^+_{H,n} \otimes_{\mathcal{O}_S} F) = p_T \circ f_F(\Lambda^+_{H,n} \otimes_{\mathcal{O}_S} F) = p_T \mathbf{D}^+_{H,n}(F)$$

which implies $f_F(\Lambda^+_{H,n} \otimes_{\mathcal{O}_S} T) = \pi_T \mathbf{D}^+_{H,n}(F)$. Since $(\tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} T) \cap \mathbf{D}^+_{H,n}(F) \subset \pi_T \mathbf{D}^+_{H,n}(F)$ we find

$$\mathbf{D}^+_{H,n}(T) \cong f_F(\Lambda^+_{H,n} \otimes_{\mathcal{O}_S} T) = (\tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} T) \cap \mathbf{D}^+_{H,n}(F).$$

Because f_F is an isomorphism we also have that $f_F(\Lambda^+_{H,n} \otimes_{\mathcal{O}_S} T) \cap f_F(\Lambda^+_{H,n} \otimes_{\mathcal{O}_S} P)$ is trivial, hence

$$\mathbf{D}^+_{H,n}(F) \cong \mathbf{D}^+_{H,n}(T) \oplus \mathbf{D}^+_{H,n}(P)$$

or, in other words, $\mathbf{D}^+_{H,n}(-)$ respects the direct sum decomposition $F = T \oplus P$.

$\mathbf{D}^+_{H,n}(T)$ is also functorial. Let $f : T \to T'$ be a G_0-equivariant morphism of free left $\mathcal{O}_S$-modules which fulfil all the requirements. Because we already know that $\mathbf{D}^+_{H,n}$ is functorial for free modules we can define $\mathbf{D}^+_{H,n}(f)$ as the composition of the top row of

$$\begin{tikzcd}
\mathbf{D}^+_{H,n}(T) \arrow[r, "\iota_T"] \arrow[d, hook] & \mathbf{D}^+_{H,n}(F) \arrow[r, "\mathbf{D}^+_{H,n}(\iota_{T'} \circ f \circ p_T)"] \arrow[d, hook] & \mathbf{D}^+_{H,n}(F') \arrow[r, "p_{T'}"] \arrow[d, hook] & \mathbf{D}^+_{H,n}(T') \arrow[d, hook] \\
\tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} T \arrow[r, "\iota_T"] & \tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} F \arrow[r, "\iota_{T'} \circ f \circ p_T"] & \tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} F' \arrow[r, "p_{T'}"] & \tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} T'
\end{tikzcd}$$

where the composition of the bottom row is just f. The functoriality of $\mathbf{D}^+_{H,n}(-)$ also implies that $\mathbf{D}^+_{H,n}(T)$ is independent of F.

$\mathbf{D}^+_{H,n}(T)$ clearly has properties (i) and (ii) since the objects and morphisms in question respect the direct sum decomposition $F = T \oplus P$ and $\mathbf{D}^+_{H,n}(F)$ has these properties. Furthermore (iii) is immediate. $\square$

We record a slight generalisation of an observation that we made in the proposition:

Corollary 4.3.2. *Let T and T' fulfil the requirements of the proposition. Then*

$$\begin{aligned}\mathbf{D}^+_{H,n}(T \oplus T') &\cong \mathbf{D}^+_{H,n}(T) \oplus \mathbf{D}^+_{H,n}(T'),\\ \mathbf{D}^+_{H,n}(T) &\cong (\tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} T) \cap \mathbf{D}^+_{H,n}(T \oplus T') \qquad \text{and}\\ \mathbf{D}^+_{H,n}(T) &= p_T \mathbf{D}^+_{H,n}(T \oplus T')\end{aligned}$$

hold, where p_T is the canonical projection $\tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} (T \oplus T') \to \tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} T$.

Proof. Let $F = T \oplus T' \oplus P$ be free, then $\mathbf{D}^+_{H,n}(X) = p_X \mathbf{D}^+_{H,n}(F) \cong (\tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} X) \cap \mathbf{D}^+_{H,n}(F)$ for $X \in \{T, T', T \oplus T'\}$ and the second and third claims follow. Regarding the first claim we just need $\mathbf{D}^+_{H,n}(X) \cong f_F(\Lambda^+_{H,n} \otimes_{\mathcal{O}_S} X)$ and that f_F is an isomorphism. $\square$

4.4 Properties of $\mathbf{D}^+_{H,n}(T)$

Lemma 4.4.1. *Let G_0' be a normal open subgroup of G_0 and assume that $\tilde{\Lambda}$ fulfils the Tate-Sen axioms for G_0 and G_0'. Let G be a subgroup of G_0' which is open and normal as a subgroup of G_0. Furthermore we assume that $(G_0', \tilde{\Lambda}, G, T, k, n)$ and $(G_0, \tilde{\Lambda}, G, \mathrm{Ind}_{G_0'}^{G_0} T, k, n)$ fulfil the requirements of proposition 4.3.1. Then*

$$\mathbf{D}^+_{H,n}(\mathrm{Ind}_{G_0'}^{G_0} T) = \mathrm{Ind}_{G_0'}^{G_0} \mathbf{D}^+_{H,n}(T).$$

Proof. Due to the direct sum decomposition of corollary 4.3.2 we may assume that T is free. Hence we would like to verify that $\mathbb{Z}[G_0] \otimes_{\mathbb{Z}[G_0']} \mathbf{D}^+_{H,n}(T)$ is the unique $\Lambda^+_{H,n}$-submodule of

$$\tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} \mathrm{Ind}_{G_0'}^{G_0} T = \tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} \left(\mathbb{Z}[G_0] \otimes_{\mathbb{Z}[G_0']} T\right) = \mathbb{Z}[G_0] \otimes_{\mathbb{Z}[G_0']} \left(\tilde{\Lambda}^+ \otimes_{\mathcal{O}_S} T\right)$$

which verifies (i)-(iii)'. (i) is clear as H is normal in G_0. (ii) is obvious. Let g_i be a set of representatives of G_0/G_0', and let d_j be a basis for $\mathbf{D}^+_{H,n}(T)$ which fulfils (iii)'. Then $g_i \otimes d_j$ is a basis of $\mathbb{Z}[G_0] \otimes_{\mathbb{Z}[G_0']} \mathbf{D}^+_{H,n}(T)$ on which G/H acts c_3-fixed[4] since G/H is central in

[4] see proposition 4.3.1(iii)

G_0/H. $\square$

Lemma 4.4.2. *Let S and S' be orthonormalisable $\mathbb{Q}_p$-Banach algebras and set $\tilde{\Lambda}_S = \mathcal{O}_S \mathbin{\widehat{\otimes}} \tilde{\Lambda}$ and $\tilde{\Lambda}_{S'} = \mathcal{O}_{S'} \mathbin{\widehat{\otimes}} \tilde{\Lambda}$. Let Y be a $\mathcal{O}_{S'}$-$\mathcal{O}_S$-bi-module which is finitely generated and projective as a topological right $\mathcal{O}_{S'}$-Banach-module. Furthermore the action of $\mathcal{O}_S$ should commute with the action of $\mathcal{O}_{S'}$ and the ring morphism $\mathcal{O}_S^{\mathrm{op}} \to \mathrm{End}_{\mathcal{O}_{S'}}(Y)$ should be continuous and of valuation ≥ 0. Let T be a finitely generated, projective left $\mathcal{O}_S$-module equipped with a continuous $\mathcal{O}_S$-linear action of G_0 and let k be an integer with $\nu_\Lambda(p^k) > c_1 + 2c_2 + 2c_3$. Assume that G is an open normal subgroup of G_0 which acts trivially on T/p^kT, set $H = G \cap H_0$ and choose $n \geq n(G)$. Then there is an isomorphism of left $\Lambda^+_{S',H,n}$-modules*

$$Y \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} \mathbf{D}^+_{H,n}(T) \xrightarrow{\sim} \mathbf{D}^+_{H,n}(Y \otimes_{\mathcal{O}_S} T).$$

Proof. We first note the following:

$$\begin{aligned} Y \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} (\tilde{\Lambda}^+_S \otimes_{\mathcal{O}_S} T) &\cong \left(Y \otimes_{\mathcal{O}_S} \widehat{\mathcal{O}_S \otimes_{\mathbb{Z}_p}} \tilde{\Lambda}^+\right) \otimes_{\mathcal{O}_S} T \\ &\cong \left(\tilde{\Lambda}^+ \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}_{S'} \otimes_{\mathcal{O}_{S'}}} Y\right) \otimes_{\mathcal{O}_S} T \\ &\cong \left(\tilde{\Lambda}^+_{S'} \otimes_{\mathcal{O}_{S'}} Y\right) \otimes_{\mathcal{O}_S} T \\ &\cong \tilde{\Lambda}^+_{S'} \otimes_{\mathcal{O}_{S'}} (Y \otimes_{\mathcal{O}_S} T). \end{aligned}$$

We assume for the moment that T and Y are *free* and we fix a (left) basis of T, Y and $\mathbf{D}^+_{H,n}(T)$, i.e. isomorphisms $b_T : \mathcal{O}_S^d \xrightarrow{\sim} T$, $b_Y : \mathcal{O}_{S'}^y \xrightarrow{\sim} Y$ and $b_D : (\Lambda^+_{S,H,n})^d \xrightarrow{\sim} \mathbf{D}^+_{H,n}(T)$.

We assume that the basis of $\mathbf{D}^+_{H,n}(T)$ chosen above fulfils condition (iii)' in proposition 4.3.1. Due to (ii), the basis is also a $\tilde{\Lambda}^+_S$-basis of $\tilde{\Lambda}^+_S \otimes_{\mathcal{O}_S} T$, i.e. there is an isomorphism $(\tilde{\Lambda}^+_S)^d \xrightarrow{\sim} \tilde{\Lambda}^+_S \otimes_{\mathcal{O}_S} T$ which does *not* come from the basis of T. In this basis $\mathbf{D}^+_{H,n}(T)$ corresponds to the image of $(\Lambda^+_{S,H,n})^d \subset (\tilde{\Lambda}^+_S)^d$:

$$\begin{array}{ccc} (\Lambda^+_{S,H,n})^d & \xrightarrow{\sim} & \mathbf{D}^+_{H,n}(T) \\ \downarrow\cap & & \downarrow\cap \\ (\tilde{\Lambda}^+_S)^d & \xrightarrow{\sim} & \tilde{\Lambda}^+_S \otimes_{\mathcal{O}_S} T. \end{array}$$

Note that

$$Y \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} \Lambda^+_{S,H,n} \cong Y \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} (\mathcal{O}_S \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} \Lambda^+_{H,n}) \cong Y \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} \Lambda^+_{H,n} \xleftarrow{\sim} (\Lambda^+_{S',H,n})^y$$

and

$$Y \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} \tilde{\Lambda}^+_S \cong Y \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} (\mathcal{O}_S \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} \tilde{\Lambda}^+) \cong Y \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} \tilde{\Lambda}^+ \xleftarrow{\sim} (\tilde{\Lambda}^+_{S'})^y$$

where the last arrows are induced by b_Y, i.e. the basis of Y. In particular, $Y \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} \Lambda^+_{S,H,n} \to Y \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} \tilde{\Lambda}^+_S$ is injective as $\Lambda^+_{S',H,n} \subset \tilde{\Lambda}^+_{S'}$ (see proposition 4.1.6). Hence:

$$\begin{tikzcd}
(\Lambda^+_{S',H,n})^{yd} \arrow[r, "\sim"] \arrow[d, hook, "\cap"] & Y \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} \mathbf{D}^+_{H,n}(T) \arrow[d, hook, "\cap"] & \\
(\tilde{\Lambda}^+_{S'})^{yd} \arrow[r, "\sim"] & Y \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} \left(\tilde{\Lambda}^+_S \otimes_{\mathcal{O}_S} T\right) \arrow[r, "\sim"] & \tilde{\Lambda}^+_{S'} \otimes_{\mathcal{O}_{S'}} (Y \otimes_{\mathcal{O}_S} T)\,,
\end{tikzcd}$$

and we are able to deduce that $Y \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} \mathbf{D}^+_{H,n}(T)$ is a free left $\Lambda^+_{S',H,n}$-submodule of $\tilde{\Lambda}^+_{S'} \otimes_{\mathcal{O}_{S'}} (Y \otimes_{\mathcal{O}_S} T)$ of rank yd.

The lemma now follows for free modules if we are able to show that $Y \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} \mathbf{D}^+_{H,n}(T)$ fulfils (i)-(iii)' from proposition 4.3.1. Indeed, (i) and (ii) are clear by construction.

Regarding (iii)', we take the basis which is induced by the isomorphism

$$\left(\Lambda^+_{S',H,n}\right)^{yd} \xrightarrow{\sim} Y \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} \left(\Lambda^+_{S,H,n}\right)^d \xrightarrow{\sim} Y \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} \mathbf{D}^+_{H,n}(T)$$

from above. Let $W_\gamma = (w_{i,j})$ be the matrix associated with $\gamma \in G/H$ when γ is acting on $\mathbf{D}^+_{H,n}(T)$ and let ψ be the ring homomorphism $\mathcal{O}_S^{\mathrm{op}} \to M_y(\mathcal{O}_{S'})$. Then it is clear that the matrix for γ acting on $Y \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} \mathbf{D}^+_{H,n}(T)$ is a block matrix W'_γ with blocks $\psi(w_{i,j})$. Hence,

$$v_{\Lambda,S'}(W'_\gamma - 1) = \min v_{\Lambda,S'}(\psi(w_{i,j}) - \delta_{i,j} I_y) \geq \min v_{\Lambda,S}(w_{i,j} - \delta_{i,j}) \geq v_{\Lambda,S}(W_\gamma - 1) > c_3$$

as ψ has valuation ≥ 0 where $\delta_{i,j}$ is the Kronecker delta.

We now assume that Y and T are projective modules and direct summands of the free modules Z and F. Then

$$\begin{tikzcd}
Z \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} (\tilde{\Lambda}_S^+ \otimes_{\mathcal{O}_S} F) \arrow[rrr, "\sim"] \arrow[dr, two heads, "p_Y \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} (\tilde{\Lambda}_S^+ \otimes_{\mathcal{O}_S} p_T)"] & & & \tilde{\Lambda}_{S'}^+ \mathbin{\widehat{\otimes}}_{\mathcal{O}_{S'}} (Z \otimes_{\mathcal{O}_S} F) \arrow[dl, two heads, "\tilde{\Lambda}_{S'}^+ \mathbin{\widehat{\otimes}}_{\mathcal{O}_{S'}} (p_Y \otimes p_T)"'] \\
& Y \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} (\tilde{\Lambda}_S^+ \otimes_{\mathcal{O}_S} T) \arrow[r, "\sim"] & \tilde{\Lambda}_{S'}^+ \mathbin{\widehat{\otimes}}_{\mathcal{O}_{S'}} (Y \otimes_{\mathcal{O}_S} T) & \\
& Y \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} \mathbf{D}_{H,n}^+(T) \arrow[u, "\cup"] \arrow[r, dashed] & \mathbf{D}_{H,n}^+(Y \otimes_{\mathcal{O}_S} T) \arrow[u, "\cup"] & \\
Z \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} \mathbf{D}_{H,n}^+(F) \arrow[uuu, "\cup"] \arrow[ur, two heads, "p_Y \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} (\tilde{\Lambda}_S^+ \otimes_{\mathcal{O}_S} p_T)"'] \arrow[rrr, "\sim"] & & & \mathbf{D}_{H,n}^+(Z \otimes_{\mathcal{O}_S} F). \arrow[uuu, "\cup"'] \arrow[ul, two heads, "\tilde{\Lambda}_{S'}^+ \mathbin{\widehat{\otimes}}_{\mathcal{O}_{S'}} (p_Y \otimes p_T)"]
\end{tikzcd}$$

where p_Y and p_T are the canonical projection morphisms $Z \to Y$ and $F \to T$ respectively and the lower diagonal morphisms are surjective by corollary 4.3.2. The commutativity of the diagram ensures that the homomorphism $Y \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} \mathbf{D}_{H,n}^+(T) \to \mathbf{D}_{H,n}^+(Y \otimes_{\mathcal{O}_S} T)$ exists and is an isomorphism. $\square$

Lemma 4.4.3. *Let S and S' be orthonormalisable $\mathbb{Q}_p$-Banach algebras and set $\tilde{\Lambda}_S = \mathcal{O}_S \mathbin{\widehat{\otimes}} \tilde{\Lambda}$ and $\tilde{\Lambda}_{S'} = \mathcal{O}_{S'} \mathbin{\widehat{\otimes}} \tilde{\Lambda}$. We assume that S is a* commutative *$\mathbb{Q}_p$-Banach algebra and that there is a continuous ring homomorphism $S \to S'$. Let T and T' be finitely generated, projective $\mathcal{O}_S$- and (left) $\mathcal{O}_{S'}$-modules respectively equipped with a continuous $\mathcal{O}_S$-linear and $\mathcal{O}_{S'}$-linear action of G_0 respectively. Furthermore, let k be an integer with $\nu_\Lambda(p^k) > c_1 + 2c_2 + 2c_3$. Assume that G is an open normal subgroup of G_0 which acts trivially on T/p^kT and T'/p^kT', set $H = G \cap H_0$ and choose $n \geq n(G)$. Then $T_{S'} := \mathcal{O}_{S'} \otimes_{\mathcal{O}_S} T$ is a free left and right $\Lambda_{S',H,n}^+$-module and there is an isomorphism of left $\Lambda_{S',H,n}^+$-modules*

$$\mathbf{D}_{H,n}^+(T_{S'}) \otimes_{\Lambda_{S',H,n}^+} \mathbf{D}_{H,n}^+(T') \xrightarrow{\sim} \mathbf{D}_{H,n}^+(T_{S'} \otimes_{\mathcal{O}_{S'}} T').$$

Proof. We first note that we can compute $\mathbf{D}_{H,n}^+(T_{S'}) \cong \mathcal{O}_{S'} \mathbin{\widehat{\otimes}}_{\mathcal{O}_S} \mathbf{D}_{H,n}^+(T)$ by considering $T_{S'}$ as a left or right representation. However, as $\mathbf{D}_{H,n}^+(T)$ is insensitive towards the left and right module distinction because S is *commutative*, the same is true for $\mathbf{D}_{H,n}^+(T_{S'})$.

We assume for the moment that T and Y are *free* and we fix a (left) basis of $\mathbf{D}_{H,n}^+(T)$ and $\mathbf{D}_{H,n}^+(T')$, i.e. isomorphisms $b_D : (\Lambda_{S',H,n}^+)^d \xrightarrow{\sim} \mathbf{D}_{H,n}^+(T_{S'})$ and $b'_D : (\Lambda_{S',H,n}^+)^{d'} \xrightarrow{\sim} \mathbf{D}_{H,n}^+(T')$. Then we have the following commutative diagram:

$$\begin{tikzcd}
(\Lambda^+_{S',H,n})^d \otimes_{\Lambda^+_{S',H,n}} (\Lambda^+_{S',H,n})^{d'} \arrow[r, "\sim"] \arrow[d, hook] & \mathbf{D}^+_{H,n}(T_{S'}) \otimes_{\Lambda^+_{S',H,n}} \mathbf{D}^+_{H,n}(T') \arrow[d] \\
(\tilde{\Lambda}^+_{S'})^d \otimes_{\tilde{\Lambda}^+_{S'}} (\tilde{\Lambda}^+_{S'})^{d'} \arrow[r, "\sim"] & \tilde{\Lambda}^+_{S'} \otimes_{\Lambda^+_{S',H,n}} \left(\mathbf{D}^+_{H,n}(T_{S'}) \otimes_{\Lambda^+_{S',H,n}} \mathbf{D}^+_{H,n}(T')\right)
\end{tikzcd}$$

where the left vertical morphism is an inclusion due to $\Lambda^+_{S',H,n} \subset \tilde{\Lambda}^+_{S'}$. Hence, the right vertical morphism is also injective and we deduce that $\mathbf{D}^+_{H,n}(T_{S'}) \otimes_{\Lambda^+_{S',H,n}} \mathbf{D}^+_{H,n}(T')$ is a free left $\Lambda^+_{S',H,n}$-submodule of

$$\begin{aligned}
&\tilde{\Lambda}^+_{S'} \otimes_{\Lambda^+_{S',H,n}} \left(\mathbf{D}^+_{H,n}(T_{S'}) \otimes_{\Lambda^+_{S',H,n}} \mathbf{D}^+_{H,n}(T')\right) \\
&\quad \xrightarrow{\sim} \left(\mathbf{D}^+_{H,n}(T_{S'}) \otimes_{\Lambda^+_{S',H,n}} \tilde{\Lambda}^+_{S'}\right) \otimes_{\tilde{\Lambda}^+_{S'}} \left(\tilde{\Lambda}^+_{S'} \otimes_{\Lambda^+_{S',H,n}} \mathbf{D}^+_{H,n}(T')\right) \\
&\quad \xrightarrow{\sim} \left(T_{S'} \otimes_{\mathcal{O}_{S'}} \tilde{\Lambda}^+_{S'}\right) \otimes_{\tilde{\Lambda}^+_{S'}} \left(\tilde{\Lambda}^+_{S'} \otimes_{\mathcal{O}_{S'}} T'\right) \\
&\quad \xrightarrow{\sim} \tilde{\Lambda}^+_{S'} \otimes_{\mathcal{O}_{S'}} \left(T_{S'} \otimes_{\mathcal{O}_{S'}} T'\right)
\end{aligned}$$

The desired statement for free modules now follows if we are able to show that the module $\mathbf{D}^+_{H,n}(T_{S'}) \otimes_{\Lambda^+_{S',H,n}} \mathbf{D}^+_{H,n}(T')$ fulfils (i)-(iii)' from proposition 4.3.1. Indeed, (i) and (ii) are clear by construction.

Regarding (iii)', we assume that the bases of $\mathbf{D}^+_{H,n}(T_{S'})$ and $\mathbf{D}^+_{H,n}(T')$ fulfil (iii)' and we take the basis which is induced by the isomorphism

$$(\Lambda^+_{S',H,n})^d \otimes_{\Lambda^+_{S',H,n}} (\Lambda^+_{S',H,n})^{d'} \xrightarrow{\sim} \mathbf{D}^+_{H,n}(T_{S'}) \otimes_{\Lambda^+_{S',H,n}} \mathbf{D}^+_{H,n}(T').$$

It is then clear that the matrix associated with γ is $W''_\gamma = W_\gamma(T_{S'}) \otimes W_\gamma(T')$, hence $v_{\Lambda,S}(W''_\gamma - 1) \geq c_3$ because

$$W_\gamma(T_{S'}) \otimes W_\gamma(T') - 1 \otimes 1 = W_\gamma(T_{S'}) \otimes (W_\gamma(T') - 1) + (W_\gamma(T_{S'}) - 1) \otimes 1$$

and we deduce that this basis also fulfils (iii)'.

We now assume that T and T' are projective modules and direct summands of the free modules F and F'. Then there is the commutative diagram

$$\begin{tikzcd}
\left(\tilde{\Lambda}^+_{S'} \otimes_{\mathcal{O}_{S'}} F_{S'}\right) \otimes_{\tilde{\Lambda}^+_{S'}} \left(\tilde{\Lambda}^+_{S'} \otimes_{\mathcal{O}_{S'}} F'\right) \arrow[rrr, "\sim"] \arrow[dr, two heads, "(\tilde{\Lambda}^+_{S'} \otimes p_T) \otimes (\tilde{\Lambda}^+_{S'} \otimes p_{T'})"] & & & \left(F_{S'} \otimes_{\mathcal{O}_{S'}} \tilde{\Lambda}^+_{S'}\right) \otimes_{\tilde{\Lambda}^+_{S'}} \left(\tilde{\Lambda}^+_{S'} \otimes_{\mathcal{O}_{S'}} F'\right) \arrow[dl, two heads, "(p_F \otimes \tilde{\Lambda}^+_{S'}) \otimes (\tilde{\Lambda}^+_{S'} \otimes p_{F'})"'] \\
& (\tilde{\Lambda}^+_{S'} \otimes_{\mathcal{O}_{S'}} T_{S'}) \otimes_{\tilde{\Lambda}^+_{S'}} (\tilde{\Lambda}^+_{S'} \otimes_{\mathcal{O}_{S'}} T') \arrow[r, "\sim"] & \tilde{\Lambda}^+_{S'} \otimes_{\mathcal{O}_{S'}} (T_{S'} \otimes_{\mathcal{O}_{S'}} T') & \\
& \mathbf{D}^+_{H,n}(T_{S'}) \otimes_{\Lambda^+_{S',H,n}} \mathbf{D}^+_{H,n}(T') \arrow[u, "\cup"] \arrow[r, dashed] & \mathbf{D}^+_{H,n}(T_{S'} \otimes_{\mathcal{O}_{S'}} T') \arrow[u, "\cup"'] & \\
\mathbf{D}^+_{H,n}(F_{S'}) \otimes_{\Lambda^+_{S',H,n}} \mathbf{D}^+_{H,n}(F') \arrow[uuu, "\cup"] \arrow[rrr, "\sim"] \arrow[ur, two heads, "(\tilde{\Lambda}^+_{S'} \otimes p_T) \otimes (\tilde{\Lambda}^+_{S'} \otimes p_{T'})"'] & & & \mathbf{D}^+_{H,n}(F_{S'} \otimes_{\mathcal{O}_{S'}} F') \arrow[uuu, "\cup"'] \arrow[ul, two heads, "(p_F \otimes \tilde{\Lambda}^+_{S'}) \otimes (\tilde{\Lambda}^+_{S'} \otimes p_{F'})"]
\end{tikzcd}$$

where p_T and $p_{T'}$ are the canonical projection morphisms $F \to T$ and $F' \to T'$ respectively and the lower diagonal morphisms are surjective by corollary 4.3.2. The commutativity of the diagram ensures that the homomorphism $\mathbf{D}^+_{H,n}(T_{S'}) \otimes_{\Lambda^+_{S',H,n}} \mathbf{D}^+_{H,n}(T') \to \mathbf{D}^+_{H,n}(T_{S'} \otimes_{\mathcal{O}_{S'}} T')$ exists and is an isomorphism. $\square$

The following lemma is an adaptation of [AI08, Prop. 7.7].

Lemma 4.4.4. *Assume that we are in a situation where (TS3) is valid and let D be a free, finitely generated $\Lambda_{H,n}$-module with an action of a γ-semi-linear operator γ'. Choose a basis (e_i) of D and denote its associated matrix by $W_{\gamma'}$. If $\nu_\Lambda(W_{\gamma'} - 1) > c_3$ holds, the module $\tilde{\Lambda}^H \otimes_{\Lambda_{H,n}} D$ decomposes as $D \oplus X_{H,n}(D)$ and $\gamma' - 1$ is continuously invertible on $X_{H,n}(D)$.*

Proof. Define $X_{H,n}(D)$ as $X_{H,n} \otimes_{\Lambda_{H,n}} D$. The claim regarding the direct sum decomposition is obvious (see remark 4.1.5) hence we are left with the continuous invertibility of $\gamma' - 1$.

As γ' fulfils $\nu_\Lambda(W_{\gamma'} - 1) > c_3$, we find an $\varepsilon > 0$ such that $\nu_\Lambda(W_{\gamma'} - 1) \geq c_3 + \varepsilon$. We have $X_{H,n}(T) = \oplus X_{H,n} e_i$ and the standard valuation on $\tilde{\Lambda}^H \otimes_{\Lambda_{H,n}} D$ restricts to the valuation $\nu_\Lambda(\sum z_i \cdot e_i) := \min\{\nu_\Lambda(z_i)\}$ on $X_{H,n}(T)$. We define the following continuous $\mathcal{O}_S$-linear automorphism f of $X_{H,n}(T)$:

$$\sum z_i \cdot e_i \mapsto \sum (\gamma - 1)^{-1}(z_i) \cdot e_i$$

which is well-defined because $\gamma - 1$ is invertible on $X_{H,n}$ by (TS3).

We also define the continuous map g_z on $X_{H,n}(T)$:

$$y \mapsto y - f\big((\gamma' - 1)(y) + z\big)$$

which is an $\mathcal{O}_S$-linear endomorphism if $z = 0$. Note that we have

$$\begin{aligned} g_0(y) &= y - f\big((\gamma' - 1)(y)\big) \\ &= \sum y_i \cdot e_i - f\big((\gamma' - 1)(y)\big) \\ &= f\big(\sum(\gamma - 1)(y_i) \cdot e_i\big) - f\big((\gamma' - 1)(y)\big) \\ &= f\big(\sum(\gamma - 1)(y_i) \cdot e_i - (\gamma' - 1)(y)\big) \\ &= f\big(\sum \gamma(y_i) \cdot e_i - \gamma'(y)\big) \\ &= f\big(\sum \gamma(y_i) \cdot e_i - \sum \gamma(y_i)\gamma'(e_i)\big) \\ &= -f\big(\sum \gamma(y_i) \cdot (\gamma' - 1)(e_i)\big). \end{aligned}$$

By (TS3), $\nu_\Lambda(f(x)) \geq \nu_\Lambda(x) - c_3$ and

$$\nu_\Lambda(g_0(y)) \geq \nu_\Lambda(y) + \min\left\{\nu_\Lambda\big((\gamma' - 1)(e_i)\big)\right\} - c_3$$

using the above computation. As noted above $\min\{\nu_\Lambda((\gamma' - 1)(e_i))\} \geq c_3 + \varepsilon$, hence

$$\nu_\Lambda(g_z(y_1) - g_z(y_2)) = \nu_\Lambda(g_0(y_1 - y_2)) \geq \nu_\Lambda(y_1 - y_2) + \varepsilon.$$

Thus, g_z is a contraction and there is a unique fixed point y_z. As f is bijective, y_z is the only solution of $-z = (\gamma' - 1)(y)$, i.e. $\gamma' - 1$ is bijective on $X_{H,n}(T)$.

We now have to check continuity, i.e. find a bound B such that $\nu_\Lambda(y_z) \geq \nu_\Lambda(z) + B$. We note that $\nu_\Lambda(y_z) \geq \min(\nu_\Lambda(z), \nu_\Lambda(y_z - z))$ as $y_z = z + (y_z - z)$. Hence, it suffices to find a bound B such that $\nu_\Lambda(y_z - z) \geq \nu_\Lambda(z) + B$:

$$\begin{aligned} \nu_\Lambda(y_z - z) &= \nu_\Lambda(y_z - g_z(z) + g_z(z) - z) \\ &\geq \min\big(\nu_\Lambda(y_z - g_z(z)), \nu_\Lambda(g_z(z) - z)\big) \\ &= \min\big(\nu_\Lambda(g_z(y_z) - g_z(z)), \nu_\Lambda(g_z(z) - z)\big) \\ &\geq \min\big(\nu_\Lambda(y_z - z) + \varepsilon, \nu_\Lambda(g_z(z) - z)\big) \\ &= \nu_\Lambda(g_z(z) - z) \\ &= \nu_\Lambda(f(\gamma'(z))) \end{aligned}$$

$$\geq \nu_\Lambda(\gamma'(z)) - c_3$$
$$\geq \nu_\Lambda(z) - c_3$$

and we deduce that the inverse of $\gamma' - 1$ is continuous. $\square$

Corollary 4.4.5. *Assume that D can be written as $D_1 \oplus D_2$, where D_i is a projective $\Lambda_{H,n}$-module, and that γ' respects the direct sum decomposition, i.e. $\gamma' = \gamma'_{D_1} \oplus \gamma'_{D_2}$ where γ'_{D_i} is the restriction of γ' to D_i. Then the module $\tilde{\Lambda}^H \otimes_{\Lambda_{H,n}} D_i$ also decomposes as $D_i \oplus X_{H,n}(D_i)$ and $\gamma'_{D_i} - 1$ is continuously invertible on $X_{H,n}(D_i)$.*

Proof. Set $X_{H,n}(D_i) \coloneqq X_{H,n} \otimes_{\Lambda_{H,n}} D_i$. The only thing left is to show that the inverse of the restriction $\gamma' - 1$ to $X_{H,n}(D_i)$ is still bijective. However this follows by two applications of the ker-coker-sequence. $\square$

5 (φ, Γ_K)-Modules with Non-Commutative Coefficients

Now we redevelop the theory of (φ, Γ_K)-modules for potentially non-commutative $\mathbb{Q}_p$-Banach algebras using the results of the last chapter. Furthermore we define and examine the cohomology of (φ, Γ_K)-modules along the lines of Pottharst's work.

5.1 Robba Ring and Related Rings

We quickly recap the definition of several rings related to the Robba ring. The precise definitions are carefully stated in various places, e.g. [Ber02] and there are many nice short introductions like [Pot13, §2.1]. Due to this wealth of resources we refrain from stating the elaborate definitions (a quick overview can be found here [Bel15, App. B]) and we just describe the results that we need.

Let $\chi : \mathrm{Gal}_K \to \mathbb{Z}_p^\times$ be the cyclotomic character with kernel H_K and image Γ_K.

There is a ring $\tilde{\mathbf{A}}^{(0,r]}$ which is separated and complete with respect to the topology induced by a valuation $\nu^{(0,r]}$. It is equipped with a Frobenius action $\varphi : \tilde{\mathbf{A}}^{(0,r]} \to \tilde{\mathbf{A}}^{(0,p^{-1}r]}$ and the Galois action is continuous. There is the subring $\tilde{\mathbf{A}}^{\dagger,s} = (\tilde{\mathbf{A}}^{(0,(p-1)/(ps)]})^+$ and we define $\tilde{\mathbf{A}}_K^{\dagger,s} := (\tilde{\mathbf{A}}^{\dagger,s})^{H_K}$. Furthermore, there is also the imperfect subring $\mathbf{A}_K^{\dagger,s}$ of $\tilde{\mathbf{A}}_K^{\dagger,s}$. We say the ring is imperfect because the Frobenius action is not surjective any more.

One goes from the rings $\mathbf{A}$ to the corresponding ring $\mathbf{B}$ by inverting p, in particular the $\mathbb{Q}_p$-Banach algebras $\mathbf{B}_K^{\dagger,s}$ and $\tilde{\mathbf{B}}_K^{\dagger,s}$ are constructed in this way.

The ring $\mathbf{B}_K^{\dagger,s}$ admits certain semi-norms and its Hausdorff completion yields a K-Fréchet-Stein algebra $\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$. The ring $\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$ contains a special element t which corresponds to $\log(1 + \pi_{\mathbb{Q}_p})$ and fulfils $\varphi(t) = p \cdot t$ and $\gamma(t) = \chi(\gamma) \cdot t$ for all $\gamma \in \Gamma_K$.

The rings without s are defined to be $\varinjlim_s$ of the rings with s, e.g. $\mathbf{B}_K^\dagger := \varinjlim_s \mathbf{B}_K^{\dagger,s}$.

A different perspective on the Robba ring $\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$ can be given via the ring $\mathcal{R}^{c_K/s}(\pi_K)$, where c_K is a constant depending on K and $\mathcal{R}^{s'}$ is defined in [KPX14, Def. 2.2.2], since the two are isomorphic albeit non-canonically for $s \gg 0$ (see also [Bel15, App. B.3] and [Ber08, §1.1]).

Definition 5.1.1. Let S be a $\mathbb{Q}_p$-Banach algebra and let $S \widehat{\otimes} \mathbf{B}_K^{\dagger,s}$ and $S \widehat{\otimes} \tilde{\mathbf{B}}_K^{\dagger,s}$ denote the usual Hausdorff completion.[1] However $S \widehat{\otimes} \mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$ denotes, by abuse of notation, $\varprojlim(S \widehat{\otimes} B_n)$ where the $\mathbb{Q}_p$-Banach algebras B_n are defined by the $\mathbb{Q}_p$-Fréchet-Stein algebra presentation $\mathbf{B}_{\mathrm{rig},K}^{\dagger,s} = \varprojlim B_n$ (for B_n see for example [KPX14, Def. 2.2.2]). Moreover we define $S \widehat{\otimes} \mathbf{B}_K^{\dagger} := \varinjlim S \widehat{\otimes} \mathbf{B}_K^{\dagger,s}$, etc. We note that we will write $\mathbf{B}_K^{\dagger,(s)}$ if both $\mathbf{B}_K^{\dagger,s}$ and $\mathbf{B}_K^{\dagger}$ are allowed.

Remark 5.1.2. Let A_n be as in §3.5, then comparing the ring $\varprojlim A_n \widehat{\otimes} \mathbf{B}_K^{\dagger,s}$ to the generalised Robba ring constructed in [Záb12] looks like a natural undertaking.

Lemma 5.1.3. *Let S be a noetherian $\mathbb{Q}_p$-Banach algebra and assume that $\mathbf{B}$ is $\mathbf{B}_K^{\dagger,(s)}$ or $\mathbf{B}_{\mathrm{rig},K}^{\dagger,(s)}$. Then $S \widehat{\otimes} \mathbf{B}$ is flat over S.*

Proof. We first note that the flatness statement for the rings without s follows from the flatness of the rings with s since direct limits of flat modules are flat.

Let $M \hookrightarrow M'$ be an injective morphism of finitely generated S-modules. We note that they are also finitely presented since S is noetherian. In the following let N be some finitely presented S-module, then:

$$(S \widehat{\otimes} \mathbf{B}_{\mathrm{rig},K}^{\dagger,(s)}) \otimes_S N = \left(\varprojlim(S \widehat{\otimes} B_n)\right) \otimes_S N \cong \varprojlim\left((S \widehat{\otimes} B_n) \otimes_S N\right)$$

where the isomorphism holds because the projective limit of inverse systems with transition maps which have dense image commutes with tensoring with finitely presented modules (see the proof of corollary 3.7.12). Furthermore, if $\mathbf{B}' \in \{\mathbf{B}_K^{\dagger,s}, B_n\}$, then

$$(S \widehat{\otimes} \mathbf{B}') \otimes_S N \cong \mathbf{B}' \widehat{\otimes} S \widehat{\otimes}_S N \cong \mathbf{B}' \widehat{\otimes} N.$$

The first isomorphism follows from the observation that $(S \widehat{\otimes} B_n) \otimes_S N$ is complete because N is finitely presented and the quotient of $\mathbb{Q}_p$-Banach algebras is again a $\mathbb{Q}_p$-Banach algebra. The second isomorphism can be deduced from corollary 2.1.15.

Since $\varprojlim$ is left exact, the statement follows if we can verify that

$$\mathbf{B}' \widehat{\otimes} M \longrightarrow \mathbf{B}' \widehat{\otimes} M'$$

is also injective. However $\mathbf{B}' \widehat{\otimes} -$ is an exact functor since $\mathbf{B}'$ is orthonormalisable (see corollary 2.3.6). ∎

[1] In the context of (φ, Γ_K)-modules we will systematically omit the base ring $\mathbb{Q}_p$ from the tensor product.

5.2 Definition of (φ, Γ_K)-Modules

Definition 5.2.1. From now on $\mathbf{B}_K^{(s)}$ denotes, for example one of the following rings: $S \widehat{\otimes} \mathbf{B}_K^{\dagger,(s)}$, $S \widehat{\otimes} \mathbf{B}_{\mathrm{rig},K}^{\dagger,(s)}$ or $S \widehat{\otimes} \tilde{\mathbf{B}}_K^{\dagger,(s)}$ where S is a $\mathbb{Q}_p$-Banach algebra. If D^s is a $\mathbf{B}_K^s$-module we define $D^{s'}$ to be the base change $\mathbf{B}_K^{s'} \otimes_{\mathbf{B}_K^s} D^s$ for $s' \geq s$ or $s' = \varnothing$.

Definition 5.2.2. A *(left) φ-module over* $\mathbf{B}_K^s$ is a finitely generated, projective left $\mathbf{B}_K^s$-module D^s equipped with a φ-semilinear morphism $\varphi : D^s \to D^{ps}$ such that the induced linear map $\varphi : \mathbf{B}_K^{ps} \otimes_{\varphi, \mathbf{B}_K^s} D^s \to D^{ps}$ is an isomorphism. A left φ-module over $\mathbf{B}_K$ is a base change of a φ-module over $\mathbf{B}_K^s$ for some s.

A *left (φ, Γ_K)-module over* $\mathbf{B}_K^{(s)}$ is a left φ-module over $\mathbf{B}_K^{(s)}$ additionally equipped with a commuting semi-linear continuous action of Γ_K.

We require morphisms between φ-modules or (φ, Γ_K)-modules to be φ-equivariant or φ and Γ_K-equivariant respectively. We denote the category of (φ, Γ_K)-modules over $S \widehat{\otimes} \mathbf{B}_K^{\dagger,(s)}$ by $\boldsymbol{\varphi}\boldsymbol{\Gamma}_{K,S}^{(s)}$. Analogously we define $\boldsymbol{\varphi}\boldsymbol{\Gamma}_{\mathrm{rig},K,S}^{(s)}$ and $\boldsymbol{\varphi}\tilde{\boldsymbol{\Gamma}}_{K,S}^{(s)}$.

Remark 5.2.3. The tensor product of two φ-modules (two (φ, Γ_K)-modules) where at least one of them is two-sided is again a φ-module ((φ, Γ_K)-module) when one employs the diagonal actions.

Remark 5.2.4. In the case S where is a finite field extension of $\mathbb{Q}_p$, $(\varphi, \Gamma_{\mathbb{Q}_p})$-modules over $S \widehat{\otimes} \mathbf{B}_{\mathrm{rig},\mathbb{Q}_p}^{\dagger,(s)}$ are automatically free since one can show that $S \widehat{\otimes} \mathbf{B}_{\mathrm{rig},\mathbb{Q}_p}^{\dagger,(s)}$ is a Bézout domain (see [Ked05, Thm. 2.9.6]).

Definition 5.2.5. We say that a φ-module D is *(c,d)-pure* ($d > 0$) if there is an integral lattice D_{int} over $\mathcal{O}_S \widehat{\otimes} \mathbf{A}_K^\dagger$ such that the induced map

$$1 \otimes \pi^c \varphi^d : (\mathcal{O}_S \widehat{\otimes} \mathbf{A}_K^\dagger) \otimes_{\varphi^d, \mathcal{O}_S \widehat{\otimes} \mathbf{A}_K^\dagger} D_{\mathrm{int}} \to D_{\mathrm{int}}$$

exists and is an isomorphism. We say that D is *étale* if the module is $(0,1)$-pure.

Definition 5.2.6. Assume that L/K is a finite Galois extension.

For a (φ, Γ_K)-module $D^{(s)}$ over $\mathbf{B}_K^{(s)}$ we define $\operatorname{Res}_K^L D^{(s)} := \mathbf{B}_L^{(s)} \otimes_{\mathbf{B}_K^{(s)}} D^{(s)}$ to be the *restricted (φ, Γ_L)-module of $D^{(s)}$*.

For a (φ, Γ_L)-module $D^{(s)}$ over $\mathbf{B}_L^{(s)}$ we define $\operatorname{Ind}_L^K D^{(s)} := \mathbb{Z}[\Gamma_K] \otimes_{\mathbb{Z}[\Gamma_L]} D^{(s)}$ to be the *induced (φ, Γ_K)-module of $D^{(s)}$*.

Next we are interested in (φ, Γ_K)-modules which can be obtained by successive extensions of rank one (φ, Γ_K)-modules.

Definition 5.2.7. Following Colmez we call a (φ, Γ_K)-module $D^{(s)}$ over $\mathbf{B}_K^{(s)}$ *triangulable* if there is an increasing (φ, Γ_K)-module filtration (which is separated and exhaustive) such that the graded pieces are free rank one (φ, Γ_K)-modules over $\mathbf{B}_K^{(s)}$.

We say $D^{(s)}$ is *potentially triangulable* if $D^{(s)}$ becomes triangulable after a finite extension of the coefficients S.

Remark 5.2.8. A triangular (φ, Γ_K)-module has the special property that there exists a basis such that the associated matrices of the actions of φ and Γ_K are triangular.

We now introduce an important class of rank one (φ, Γ_K)-modules: the (φ, Γ_K)-modules of character type and we simplify the discussion by assuming that $K = \mathbb{Q}_p$. The full story can be found in [KPX14, §6.2].

Definition 5.2.9. Let S be commutative. For any continuous character $\delta : \mathbb{Q}_p^\times \to S^\times$ there exists a free, rank one $(\varphi, \Gamma_{\mathbb{Q}_p})$-module $\mathbf{B}_{\mathbb{Q}_p}^{(s)}(\delta) \cdot \mathbf{e}$ over $\mathbf{B}_{\mathbb{Q}_p}^{(s)}$ which is unique up to a unique isomorphism defined by $\varphi(\mathbf{e}) = \delta(p) \cdot \mathbf{e}$ and $\gamma(\mathbf{e}) = \delta(\chi(\gamma)) \cdot \mathbf{e}$. We denote the $(\varphi, \Gamma_{\mathbb{Q}_p})$-module by $\mathbf{B}_{\mathbb{Q}_p}^{(s)}(\delta)$.

We define the *twist* $D^{(s)}(\delta)$ of a $(\varphi, \Gamma_{\mathbb{Q}_p})$-module $D^{(s)}$ to be $\mathbf{B}_K^{(s)}(\delta) \otimes_{\mathbf{B}_K^{(s)}} D^{(s)}$ which is again a $(\varphi, \Gamma_{\mathbb{Q}_p})$-module.

Remark 5.2.10. There are the following important continuous characters: $\delta(x) = |x|_p$ and $\delta(x) = x$. The Tate twist corresponds to the character $\delta(x) = x|x|_p$.

Theorem 5.2.11 ([KPX14, Thm. 6.2.14])**.** *Let A be a (commutative) affinoid and D a rank one $(\varphi, \Gamma_{\mathbb{Q}_p})$-module over $A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}$. Then D is isomorphic to $(A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p})(\delta) \otimes_A \mathcal{L}$ where $\mathcal{L}$ is an A-module of rank one and δ is a character $\mathbb{Q}_p^\times \to A^\times$.*

Remark 5.2.12. It would be interesting to see how the analogue of the theorem looks in the non-commutative situation as the proof uses some class field theory as input.

The next proposition deals with a particular problem in cohomology and is a natural extension of [Ked08, Prop. 1.2.6].

Proposition 5.2.13. *Let A be a $\mathbb{Q}_p$-nc-affinoid algebra and D^s a* free, étale *φ-module over $A \widehat{\otimes} \mathbf{B}_K^{\dagger,(s)}$ and let D^s_{rig} be the base change $(A \widehat{\otimes} \mathbf{B}^{\dagger,s}_{\mathrm{rig},K}) \otimes_{A \widehat{\otimes} \mathbf{B}_K^{\dagger,s}} D^s$. Then*

$$\mathrm{id} - \varphi : D^s_{\mathrm{rig}}/D^s \longrightarrow D^{ps}_{\mathrm{rig}}/D^{ps}$$

is an isomorphism for $s \gg 0$.

Proof. Choose a basis of D^s such that the operator φ can be represented by a matrix $F \in M_n^{\mathrm{op}}(\mathcal{O}_A \mathbin{\widehat{\otimes}} A_K^{\dagger,ps})$. Then the proposition is equivalent to

$$\begin{aligned} \mathrm{id} - \varphi \cdot F : (A \mathbin{\widehat{\otimes}} \mathbf{B}^{\dagger,s}_{\mathrm{rig},K})^n / (A \mathbin{\widehat{\otimes}} \mathbf{B}^{\dagger,s}_K)^n &\longrightarrow (A \mathbin{\widehat{\otimes}} \mathbf{B}^{\dagger,ps}_{\mathrm{rig},K})^n / (A \mathbin{\widehat{\otimes}} \mathbf{B}^{\dagger,ps}_K)^n \\ v &\longmapsto v - \varphi(v) \cdot F \end{aligned}$$

being an isomorphism.

Since there are more tools available for the ring $\mathcal{R}^s = \mathcal{R}^s(\pi_K)$ we reformulate the statement in the language of the rings $\mathcal{R}^s$, $\mathcal{R}^s_{\mathrm{bd}}$ and $\mathcal{R}^s_{\mathrm{int}}$. Their definitions can be found in [Ked08] and [KPX14]. For example $\mathcal{R}^s$ is the ring of Laurent series in the variable T with coefficients in the maximal unramified extension of the cyclotomic extension K_∞ which converge on the annulus $\omega^s \le |T| < 1$, where $\omega = p^{-1/(p-1)} < 1$. Then the proposition is for a different s equivalent to

$$\begin{aligned} \mathrm{id} - \varphi \cdot F : (A \mathbin{\widehat{\otimes}} \mathcal{R}^s)^n / (A \mathbin{\widehat{\otimes}} \mathcal{R}^s_{\mathrm{bd}})^n &\longrightarrow (A \mathbin{\widehat{\otimes}} \mathcal{R}^{s/p})^n / (A \mathbin{\widehat{\otimes}} \mathcal{R}^{s/p}_{\mathrm{bd}})^n \\ v &\longmapsto v - \varphi(v) \cdot F \end{aligned}$$

being an isomorphism for $F \in M_n^{\mathrm{op}}(\mathcal{O}_A \mathbin{\widehat{\otimes}} \mathcal{R}^{s/p}_{\mathrm{int}})$.

Since A is an nc-affinoid, there is a map $p : T_m^{nc} \twoheadrightarrow A$ with kernel I. By corollary 2.3.4, we find a continuous section s of the projection map with norm 1. Define $\tilde{F}$ as the image of F under the map $s \mathbin{\widehat{\otimes}} \mathrm{id}$ in $M_n^{\mathrm{op}}(\mathcal{O}_{T_m^{nc}} \mathbin{\widehat{\otimes}} \mathcal{R}^{s/p}_{\mathrm{int}})$.

Recall that instances of the B_n in definition 5.1.1 are the rings $\mathcal{R}^{[r_n,s]}$ where r_n is a zero-sequence. Furthermore $(M \mathbin{\widehat{\otimes}} \mathcal{R}^s)/(M \mathbin{\widehat{\otimes}} \mathcal{R}^s_{\mathrm{bd}})$ is isomorphic to the quotient $M \mathbin{\widehat{\otimes}} (\mathcal{R}^s/\mathcal{R}^s_{\mathrm{bd}})$, where the last term is defined as in definition 5.1.1, for any orthonormalisable $\mathbb{Q}_p$-Banach space M because

$$0 \longrightarrow M \mathbin{\widehat{\otimes}} \mathcal{R}^s_{\mathrm{bd}} \longrightarrow M \mathbin{\widehat{\otimes}} \mathcal{R}^{[r_n,s]} \longrightarrow M \mathbin{\widehat{\otimes}} (\mathcal{R}^{[r_n,s]}/\mathcal{R}^s_{\mathrm{bd}}) \longrightarrow 0$$

is exact due to corollary 2.3.6 and because the constant system $M \mathbin{\widehat{\otimes}} \mathcal{R}^s_{\mathrm{bd}}$ fulfils the Mittag-Leffler property. Note that in particular T_m^{nc}, I and A are orthonormalisable $\mathbb{Q}_p$-Banach modules.

Then there is the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \left(I \mathbin{\widehat{\otimes}} (\mathcal{R}^s/\mathcal{R}^s_{\mathrm{bd}})\right)^n \arrow[r] \arrow[d, "\mathrm{id} - \varphi \cdot \tilde{F}"] & \left(T_m^{nc} \mathbin{\widehat{\otimes}} (\mathcal{R}^s/\mathcal{R}^s_{\mathrm{bd}})\right)^n \arrow[r] \arrow[d, "\mathrm{id} - \varphi \cdot \tilde{F}"] & \left(A \mathbin{\widehat{\otimes}} (\mathcal{R}^s/\mathcal{R}^s_{\mathrm{bd}})\right)^n \arrow[r] \arrow[d, "\mathrm{id} - \varphi \cdot F"] & 0 \\ 0 \arrow[r] & \left(I \mathbin{\widehat{\otimes}} (\mathcal{R}^{s/p}/\mathcal{R}^{s/p}_{\mathrm{bd}})\right)^n \arrow[r] & \left(T_m^{nc} \mathbin{\widehat{\otimes}} (\mathcal{R}^{s/p}/\mathcal{R}^{s/p}_{\mathrm{bd}})\right)^n \arrow[r] & \left(A \mathbin{\widehat{\otimes}} (\mathcal{R}^{s/p}/\mathcal{R}^{s/p}_{\mathrm{bd}})\right)^n \arrow[r] & 0. \end{tikzcd}$$

Now, by the ker-coker sequence the desired statement is equivalent to the following:

(i) the second vertical morphism is injective, and

(ii) the first and second vertical morphisms are surjective.

We need some preparatory statements such as a characterisation of $T_m^{nc} \mathbin{\widehat{\otimes}} \mathcal{R}^s_{\mathrm{bd}}$ in $T_m^{nc} \mathbin{\widehat{\otimes}} \mathcal{R}^s$. As $\mathcal{R}^s_{\mathrm{bd}} \subset \mathcal{R}^s$ it consists just of the bounded elements, i.e.

$$f \in \mathcal{R}^s_{\mathrm{bd}} \Leftrightarrow f \in \mathcal{R}^s \text{ and } \exists C \text{ such that } \forall r \in (0, s] : |f|_r < C$$

where we use the Gauß norms defined in [KPX14, Not. 2.1.1]. Assuming that we have chosen a Schauder basis $(e_a)_{a \in J}$ for T_m^{nc}, then $T_m^{nc} \mathbin{\widehat{\otimes}} \mathcal{R}^s \cong \varprojlim c_J(\mathcal{R}^{[r_n, s]}) \cong c_J(\mathcal{R}^s)$, hence an element $f \in T_m^{nc} \mathbin{\widehat{\otimes}} \mathcal{R}^s$ can be represented by $(f^{(a)})_a \in c_J(\mathcal{R}^s)$ such that for all $0 < r \leq s$ we have $|f^{(a)}|_r \to 0$ for $a \to \infty$ (see section 2.3). Additionally, we find

$$(f^{(a)})_a \in c_J(\mathcal{R}^s_{\mathrm{bd}}) \Leftrightarrow (f^{(a)})_a \in c_J(\mathcal{R}^s) \text{ and } \forall a \, \exists C_a \, \forall r \in (0, s] : |f^{(a)}|_r < C_a.$$

Furthermore, we will need later on that $|\tilde{F}|_r \leq 1$ for $0 < r \leq s/p$. Let $\tilde{F} = (f_{i,j})_{i,j}$; then we have $|f^{(a)}_{i,j}|_r \to 0$ for $a \to \infty$, hence there is a C such that $|f^{(a)}_{i,j}|_{s/p} \leq C$ for all i, j and a on the boundary. We note that the claims are invariant under replacing v and $\tilde{F}$ by $T^{-u}v$ and $\varphi(T^u)T^{-u}\tilde{F}$ respectively. We always have $|\varphi(T)/T^p - 1|_r < 1$, hence

$$\begin{aligned} |\varphi(T)/T|_r &\leq |\varphi(T)/T^p|_r \cdot |T^{p-1}|_r \\ &\leq \max\{|\varphi(T)/T^p - 1|_r, |1|_r\} \cdot |T^{p-1}|_r \\ &\leq \omega^{r(p-1)} \\ &< 1. \end{aligned}$$

We assume that u is chosen in a way such that the new $\tilde{F}$ fulfils $|\tilde{F}|_{s/p} \leq 1$.

Since the entries of $\tilde{F}$ are in $\mathcal{O}_{T_m^{nc}} \mathbin{\widehat{\otimes}} \mathcal{R}^{s/p}_{\mathrm{int}}$, it follows that $|\tilde{F}|_r \leq 1$ for $0 < r < s/p$ because of the following reason: let us assume $f \in \mathcal{R}^{s'}_{\mathrm{int}}$ with $|f|_{s'} \leq 1$. Then $f = \sum_{k \in \mathbb{Z}} a_k T^k$ with $|a_k| \leq 1$ can be split in $f = f^+ + f^-$ with $f^+ = \sum_{k \geq 0} a_k T^k$ and $f^- = \sum_{k<0} a_k T^k$. It is clear that $|f^+|_r \leq 1$ holds for all $0 < r \leq s'$. Furthermore, for $0 < r \leq s'$ we find $|f^-|_r \leq |f^-|_{s'} \leq |f|_{s'}$ just because of the nature of the exponential function. We deduce that $|f|_r = \max\{|f^+|_r, |f^-|_r\} \leq 1$ as claimed.

There is a preferred basis on T_m^{nc}, namely $(X^J)_J$ with $J \in \mathbb{N}^m$. Let $|J|$ be the sum of the entries of J. We choose an enumeration of $\mathbb{N}^m$ such that $|J_a| \leq |J_b|$ for $a \leq b$ and we choose $(s_a \coloneqq X^{J_a})_a$ as our Schauder basis for T_m^{nc}. Its defining property is that its basis elements are ordered by degree. In particular, $s_a \cdot s_b$ has no non-trivial s_c component if $\max(a, b) > c$.

Now we can deal with the injectivity statement, i.e. let $v \in (T_m^{nc} \widehat{\otimes} \mathcal{R}^s)^n$ such that

$$w := v - \varphi(v) \cdot \tilde{F} \in (T_m^{nc} \widehat{\otimes} \mathcal{R}^{s/p})^n$$

is actually an element of $(T_m^{nc} \widehat{\otimes} \mathcal{R}_{\mathrm{bd}}^{s/p})^n$. We have to show that v is an element of $(T_m^{nc} \widehat{\otimes} \mathcal{R}_{\mathrm{bd}}^{s})^n$.

We want to use the preferred Schauder basis on T_m^{nc} as defined above. As we assumed that $w = (w^{(a)})_a$ is in $(T_m^{nc} \widehat{\otimes} \mathcal{R}_{\mathrm{bd}}^{s/p})^n$, there are C_a such that $|w^{(a)}|_r \leq C_a$ for $0 < r < s/p$.

Directly from the definition of $|{-}|_r$ we see

$$|v^{(a)}|_r \leq |v^{(a)}|_{s/p} + |v^{(a)}|_s$$

for $s/p \leq r \leq s$, in particular $|v^{(a)}|_r$ is bounded above in this interval. We possibly enlarge C_a such that $\max_{s/p \leq r \leq s} |v^{(a)}|_r \leq C_a$ holds. We further enlarge C_a by additionally requiring $C_a = \max_{b \leq a}\{C_b\}$.

Then, assuming $s/p^2 \leq r \leq s/p$,

$$\begin{aligned}
|v^{(a)}|_r &= |w^{(a)} + (\varphi(v) \cdot \tilde{F})^{(a)}|_r \\
&\leq \max\{|w^{(a)}|_r, |(\varphi(v) \cdot \tilde{F})^{(a)}|_r\} \\
&\overset{(1)}{\leq} \max\{C_a, \max_{b,c \leq a}\{|\varphi(v)^{(b)}|_r \cdot |\tilde{F}^{(c)}|_r\}\} \\
&\leq \max\{C_a, \max_{b \leq a}\{|\varphi(v^{(b)})|_r \cdot 1\}\} \\
&\overset{(2)}{\leq} \max\{C_a, \max_{b \leq a}\{|v^{(b)}|_{rp}\}\} \\
&\leq \max\{C_a, \max_{b \leq a}\{C_b\}\} \\
&\overset{(3)}{=} C_a
\end{aligned}$$

where (1) is true because of the special Schauder basis and (2) is true due to $|f|_r = |\varphi(f)|_{r/p}$. Furthermore, (3) holds because we assumed C_a to have this property.

We conclude, that

$$|v^{(a)}|_r \leq C_a$$

for $s/p^2 \leq r \leq s$. By iterating this argument we see that this inequality actually holds for the full interval $0 < r \leq s$, hence we deduce that v is indeed an element of $(T_m^{nc} \widehat{\otimes} \mathcal{R}_{\mathrm{bd}}^s)^n$.

Lastly we want to deduce the surjectivity statements, i.e. assume that $I \subseteq T_m^{nc}$ is a closed orthonormalisable ideal for which we choose some Schauder basis. We show that

$$\mathrm{id} - \varphi \cdot \tilde{F}: \quad (I \widehat{\otimes} (\mathcal{R}^s / \mathcal{R}_{\mathrm{bd}}^s))^n \to \left(I \widehat{\otimes} (\mathcal{R}^{s/p} / \mathcal{R}_{\mathrm{bd}}^{s/p})\right)^n$$

is surjective.

Let $w = (w^{(a)})_a \in \left(I \widehat{\otimes} \mathcal{R}^{s'}\right)^n$ be some element with $s' := s/p$. We define a sequence $(w_k)_k$ inductively as follows: start with $w_0 = w$. Assume $w_k^{(a)} = \sum_{i \in \mathbb{Z}} a_{k,i}^{(a)} T^i$, then we define $w_k^+ = \left(\sum_{i>0} a_{k,i}^{(a)} T^i\right)_a$ and $w_k^- = \left(\sum_{i \leq 0} a_{k,i}^{(a)} T^i\right)_a$. We set $w_{k+1} := \varphi(w_k^+) \cdot \tilde{F}$. Since w_k^+ has non-vanishing coefficients only in positive degrees, we see from the definition that

$$|w_k^+|_{s'} = |T|_{s'} \cdot \left|\frac{w_k^+}{T}\right|_{s'} \leq |T|_{s'} \cdot \left|\frac{w_k^+}{T}\right|_{s'/p} = \frac{|T|_{s'}}{|T|_{s'/p}} \cdot |w_k^+|_{s'/p} \leq \omega^{s' - s'/p} \cdot |w_k|_{s'/p},$$

hence $|w_{k+1}|_{s'/p} \leq \omega^{s'-s'/p} |w_k|_{s'/p}$ where we used $|\tilde{F}|_r \leq 1$ from above. We may deduce that $|w_k^+|_{s'/p}$ converges to 0, thus also $|w_k^+|_r$ converges to 0 for $s'/p \leq r$ without any upper limit for r by the nature of the Gauß norm and because w_k^+ has non-vanishing coefficients only for positive powers. The same reasoning yields for $0 < r \leq s'/p$ the following inequality:

$$|w_k^-|_r \leq |w_k^-|_{s'/p} \leq |w_k|_{s'/p}.$$

We now set $v := \sum w_k^+$ and note that v is an element of $\left(I \widehat{\otimes} \mathcal{R}^{[s'/p, \infty]}\right)^n$, hence in particular an element of $\left(I \widehat{\otimes} \mathcal{R}^{[s'/p, s]}\right)^n$. Furthermore, the right hand side of

$$w - v + \varphi(v) \cdot \tilde{F} = \sum_k w_k^-$$

is bounded on $0 < r \leq s'/p$, i.e. an element of $\left(I \widehat{\otimes} \mathcal{R}_{\mathrm{bd}}^{s'/p}\right)^n$. We also see that $\varphi(v)$ is an element of $\left(I \widehat{\otimes} \mathcal{R}^{[s'/p^2, s'/p]}\right)^n$, hence

$$v = w + \varphi(v) \cdot \tilde{F} - \sum_k w_k^-,$$

is also an element of $\left(I \widehat{\otimes} \mathcal{R}^{[s'/p^2, s'/p]}\right)^n$. Inductively we may conclude that v is indeed an element of $(I \widehat{\otimes} \mathcal{R}^s)^n$. $\square$

Corollary 5.2.14. *If D^s as in the proposition has a direct sum decomposition $D^s = D_1^s \oplus D_2^s$ in the category of φ-modules, then*

$$\mathrm{id} - \varphi : D_{i,\mathrm{rig}}^s / D_i^s \longrightarrow D_{i,\mathrm{rig}}^{ps} / D_i^{ps}$$

is also an isomorphism for $s \gg 0$.

Proof. The statement follows from the proposition and two applications of the ker-coker-sequence. □

5.3 Étale Descent

In this section we generalise [BC08, Prop. 2.2.1] to the non-commutative setting.

Let B be a commutative $\mathbb{Q}_p$-Banach algebra equipped with a continuous action of a finite group G. Let $B^\natural$ the same ring B with trivial G-action. We assume that the following holds:

(i) the B^G-module B is free of finite rank and is faithfully flat, and

(ii) we have $B^\natural \otimes_{B^G} B \cong \oplus_{g \in G} B^\natural e_g$ with $e_g e_h = \delta_{g,h} e_g$ and $g(e_h) = e_{gh}$ where $\delta_{g,h}$ is the usual δ-distribution.

Then:

Proposition 5.3.1. *If S is a $\mathbb{Q}_p$-Banach algebra with trivial G-action and M is a finitely generated, projective $S \widehat{\otimes} B$-module equipped with a semi-linear action of G, then*

(i) M^G is a finitely generated, projective $S \widehat{\otimes} B^G$-module, and

(ii) the natural map $(S \widehat{\otimes} B) \otimes_{S \widehat{\otimes} B^G} M^G \to M$ is an isomorphism.

Proof. Let $\pi_G = (\#G)^{-1} \sum_{g \in G} g \in B[G]$. For any $B[G]$-module N, there is the B^G-decomposition $N = \pi_G N \oplus \ker L_{\pi_G}$ where L_{π_G} is the left multiplication with π_G and $N^G = \pi_G N$. In particular, $M = M^G \oplus \ker L_{\pi_G}$ which shows that M^G is a finitely generated, projective $S \widehat{\otimes} B^G$-module as M is a finitely generated, projective $S \widehat{\otimes} B^G$-module due to condition (i).

Because of the isomorphism $(S \widehat{\otimes} B) \otimes_{S \widehat{\otimes} B^G} M^G \cong B \otimes_{B^G} M^G$ (see corollary 2.2.6) it suffices to show that $B \otimes_{B^G} M^G \to M$ is an isomorphism. As here the potential non-commutative nature of S does not play a role any more, the proof of [BC08, Prop. 2.2.1(ii)] can be copied. □

Corollary 5.3.2. *The inverse of the functor $(-)^G$ from the category of finitely generated, projective left $S \widehat{\otimes} B$-modules with semi-linear action of G to finitely generated, projective left $S \widehat{\otimes} B^G$-modules is the functor $(S \widehat{\otimes} B) \otimes_{S \widehat{\otimes} B^G} -$.*

Proof. Indeed

$$N \xrightarrow{\sim} \left((S \widehat{\otimes} B) \otimes_{S \widehat{\otimes} B^G} N\right)^G$$

because G is finite and

$$(S \widehat{\otimes} B) \otimes_{S \widehat{\otimes} B^G} M^G \xrightarrow{\sim} M$$

as above. $\square$

We formalise a method which is often used in [Bel15, App. D]:

Lemma 5.3.3. *Let V be a finitely generated, projective left $S \widehat{\otimes} \mathbf{B}_L^{\dagger,s}$-module and W a finitely generated, projective left $S \widehat{\otimes} \mathbf{B}_K^{\dagger,s}$-module. Then*

$$V^{H_K} \to W$$

is an isomorphism if and only if the H_K/H_L-equivariant map

$$V \to \left(S \widehat{\otimes} \mathbf{B}_L^{\dagger,s}\right) \otimes_{S \widehat{\otimes} \mathbf{B}_K^{\dagger,s}} W$$

is an isomorphism, assuming $s \geq s(L/K)$ where $s(L/K)$ is defined in [BC08, Lem. 4.2.5].

Proof. This follows from corollary 5.3.2 and [BC08, Lem. 4.2.5] where one uses that H_K/H_L is finite. $\square$

5.4 $\mathrm{D}_K^{\dagger,s}(V)$ with Non-Commutative Coefficients

We assume the following:

Hypothesis 5.4.1. S is an orthonormalisable $\mathbb{Q}_p$-Banach algebra, V a finitely generated, projective left S-module equipped with a continuous S-linear action of Gal_K and T a Galois stable projective $\mathcal{O}_S$-lattice. Furthermore, let L be a finite Galois extension of K such that Gal_L acts trivially on $T/12pT$.

Remark 5.4.2. The ideas presented in [Liu15, §1.1] mostly carry over to our situation, in particular the requirement that the lattice T is supposed to be Gal_K-stable can be weakened. However we do not pursue this path.

Proposition 5.4.3 ([BC08, Prop. 4.2.1]). *The ring $\tilde{\Lambda} = \tilde{\mathbf{A}}^{(0,1]}$ verifies (TS1), (TS2) and (TS3) with $\tilde{\Lambda}^{H_L} =: \tilde{\mathbf{A}}_L^{(0,1]}$, $\Lambda_{H_L,n} = \varphi^{-n}(\mathbf{A}_L^{(0,p^{-n}]})$, $R_{H_L,n} = R_{L,n}$ and $\nu_\Lambda = \nu^{(0,1]}$ and the constants $c_1 > 0$, $c_2 > 0$ and $c_3 > 1/(p-1)$ can be chosen arbitrarily.*

Set $\mathbf{A}_{L,n}^{\dagger,s} \coloneqq \varphi^{-n}(\mathbf{A}_L^{\dagger,p^n s})$, then there is the following

Proposition 5.4.4. *Assume that we are in situation 5.4.1 and let $n \geq n(L) \coloneqq n(\mathrm{Gal}_L)$. Then $(\mathcal{O}_S \mathbin{\widehat{\otimes}} \tilde{\mathbf{A}}^{\dagger,(p-1)/p}) \otimes_{\mathcal{O}_S} T$ possesses a functorial left $\mathcal{O}_S \mathbin{\widehat{\otimes}} \mathbf{A}_{L,n}^{\dagger,(p-1)/p}$-module, denoted by $\mathbf{D}_{L,n}^{\dagger,(p-1)/p}(T)$, which is fixed by H_L and stable under Gal_K, together with a c_3-Γ_L-invariant projective basis such that*

$$(\mathcal{O}_S \mathbin{\widehat{\otimes}} \tilde{\mathbf{A}}^{\dagger,(p-1)/p}) \otimes_{\mathcal{O}_S \mathbin{\widehat{\otimes}} \mathbf{A}_{L,n}^{\dagger,(p-1)/p}} \mathbf{D}_{L,n}^{\dagger,(p-1)/p}(T) \xrightarrow{\sim} (\mathcal{O}_S \mathbin{\widehat{\otimes}} \tilde{\mathbf{A}}^{\dagger,(p-1)/p}) \otimes_{\mathcal{O}_S} T.$$

If T is free, then $\mathbf{D}_{L,n}^{\dagger,(p-1)/p}(T)$ is also free of the same rank and there exists a c_3-Γ_L-invariant basis. Furthermore in this case $\mathbf{D}_{L,n}^{\dagger,(p-1)/p}(T)$ is the unique module with these properties.

Proof. This follows from proposition 4.3.1 as proposition 5.4.3, proposition 4.1.6 and remark 4.2.8 imply the necessary conditions. □

Corollary 5.4.5. *In the situation of the above proposition:*

(i) We have

$$\mathbf{D}_{L,n+1}^{\dagger,(p-1)/p}(T) = \left(\mathcal{O}_S \mathbin{\widehat{\otimes}} \mathbf{A}_{L,n+1}^{\dagger,(p-1)/p}\right) \otimes_{\mathcal{O}_S \mathbin{\widehat{\otimes}} \mathbf{A}_{L,n}^{\dagger,(p-1)/p}} \mathbf{D}_{L,n}^{\dagger,(p-1)/p}(T).$$

(ii) For L and $n(L)$ sufficiently large we find

$$\mathbf{D}_{L,n+1}^{\dagger,(p-1)/p}(T) = \varphi^{-1}\left((\mathcal{O}_S \mathbin{\widehat{\otimes}} \mathbf{A}_{L,n}^{\dagger,p-1}) \otimes_{\mathcal{O}_S \mathbin{\widehat{\otimes}} \mathbf{A}_{L,n}^{\dagger,(p-1)/p}} \mathbf{D}_{L,n}^{\dagger,(p-1)/p}(T)\right).$$

Proof. In the free case it follows from uniqueness (see [Liu15, Cor. 1.1.2]). In the general case we note that the constructions on the right hand side commute with projections, hence the statement follows from corollary 4.3.2. □

Definition 5.4.6. Define $r_n = (p-1)p^{n-1}$ and $s(V) = \max(r_{n(L)}, s(L/K))$. We assume (by potentially enlarging $s(V)$) that there is an integer $n(V)$ such that $r_{n(V)} = s(V)$.

Furthermore we assume that L and $n(L)$ are large enough such that corollary 5.4.5(ii) is applicable. For $s \geq s(V)$ we set

$$\mathbf{D}_K^{\dagger,s}(V) = ((S \widehat{\otimes} \mathbf{B}_L^{\dagger,s}) \otimes_{\mathcal{O}_S \widehat{\otimes} \mathbf{A}_L^{\dagger,s(V)}} \varphi^{n(V)}(\mathbf{D}_{L,n(V)}^{\dagger,(p-1)/p}(T)))^{H_K}$$

which is equipped with an action of Γ_K.

Then we deduce from the previous proposition:

Proposition 5.4.7. *Assume that we are in situation 5.4.1 and $s \geq s(V)$. Then*

(i) $\mathbf{D}_K^{\dagger,s}(V)$ *is well-defined and independent of L, T and $n(V)$ (after potentially enlarging $s(V)$),*

(ii) $\mathbf{D}_K^{\dagger,s}(V)$ *is a projective left $S \widehat{\otimes} \mathbf{B}_K^{\dagger,s}$-module (of the same rank as V, if defined), if additionally $K = L$ holds and V is free, then $\mathbf{D}_K^{\dagger,s}(V)$ is also free,*

(iii) the natural Galois-equivariant map

$$(S \widehat{\otimes} \tilde{\mathbf{B}}^{\dagger,s}) \otimes_{S \widehat{\otimes} \mathbf{B}_K^{\dagger,s}} \mathbf{D}_K^{\dagger,s}(V) \to (S \widehat{\otimes} \tilde{\mathbf{B}}^{\dagger,s}) \otimes_S V$$

is an isomorphism,

(iv) $\mathbf{D}_K^{\dagger,s}(V)$ *is compatible with finite field extensions K'/K*

$$\mathbf{D}_{K'}^{\dagger,s}(V) = \left(S \widehat{\otimes} \mathbf{B}_{K'}^{\dagger,s}\right) \otimes_{S \widehat{\otimes} \mathbf{B}_K^{\dagger,s}} \mathbf{D}_K^{\dagger,s}(V),$$

(v) $\mathbf{D}_K^{\dagger,s}(V)$ *is compatible with enlarging s, i.e. if $s' \geq s$ then*

$$\mathbf{D}_K^{\dagger,s'}(V) = \left(S \widehat{\otimes} \mathbf{B}_K^{\dagger,s'}\right) \otimes_{S \widehat{\otimes} \mathbf{B}_K^{\dagger,s}} \mathbf{D}_K^{\dagger,s}(V),$$

and

(vi) the formation of $\mathbf{D}_K^{\dagger,s}(-)$ is functorial in a natural way which is compatible with the isomorphism in (iii).

Proof. Let $n(s)$ the biggest integer n with $r_n \leq s$. Using corollary 5.4.5(ii) we deduce that

$$(S \widehat{\otimes} \mathbf{B}_L^{\dagger,s}) \otimes_{S \widehat{\otimes} \mathbf{A}_L^{\dagger,r_{n_1}}} \varphi^{n_1}(\mathbf{D}_{L,n_1}^{\dagger,(p-1)/p}(T)) = (S \widehat{\otimes} \mathbf{B}_L^{\dagger,s}) \otimes_{S \widehat{\otimes} \mathbf{A}_L^{\dagger,r_{n_2}}} \varphi^{n_2}(\mathbf{D}_{L,n_2}^{\dagger,(p-1)/p}(T))$$

as long as $n(V) \leq n_i \leq n(s)$ for $i = 1, 2$. Hence, the definition of $\mathbf{D}_K^{\dagger,s}(V)$ is insensitive with respect to replacing $n(V)$ by some n such that $n(V) \leq n \leq n(s)$.

Regarding the independence of L it suffices to check that L and L' yield the same $\mathbf{D}_K^{\dagger,s}(V)$ assuming $L \subset L'$. Proposition 5.4.4 implies that $\mathbf{D}_{L,n(V)}^{\dagger,(p-1)/p}(T)$ and $\mathbf{D}_{L',n(V)}^{\dagger,(p-1)/p}(T)$ are finitely generated projective left modules, hence due to lemma 5.3.3 it suffices to check that

$$(S \widehat{\otimes} \mathbf{B}_{L'}^{\dagger,s}) \otimes_{S \widehat{\otimes} \mathbf{A}_L^{\dagger,s(V)}} \varphi^{n(V)}(\mathbf{D}_{L,n(V)}^{\dagger,(p-1)/p}(T)) \cong (S \widehat{\otimes} \mathbf{B}_{L'}^{\dagger,s}) \otimes_{S \widehat{\otimes} \mathbf{A}_{L'}^{\dagger,s(V)}} \varphi^{n(V)}(\mathbf{D}_{L',n(V)}^{\dagger,(p-1)/p}(T))$$

is an isomorphism, which is equivalent to

$$\mathbf{D}_{L',n(V)}^{\dagger,(p-1)/p}(T) \cong (S \widehat{\otimes} \mathbf{A}_{L'}^{\dagger,n(V)}) \otimes_{S \widehat{\otimes} \mathbf{A}_L^{\dagger,n(V)}} \mathbf{D}_{L,n(V)}^{\dagger,(p-1)/p}(T)$$

being an isomorphism. This follows directly from the uniqueness statement in proposition 5.4.4 in the case T is free and the general case follows from corollary 4.3.2 since the constructions commute with projections.

The independence from T can be proved as in the commutative free case, cf. [Liu15, p. 12], and (i) is shown.

As mentioned before proposition 5.4.4 implies that $\mathbf{D}_L^{\dagger,s}(V)$ is a projective left $S \widehat{\otimes} \mathbf{B}_L^{\dagger,s}$-module and that the morphism

$$(S \widehat{\otimes} \tilde{\mathbf{B}}^{\dagger,s}) \otimes_{S \widehat{\otimes} \mathbf{B}_L^{\dagger,s}} \mathbf{D}_L^{\dagger,s}(V) \to (S \widehat{\otimes} \tilde{\mathbf{B}}^{\dagger,s}) \otimes_S V$$

is an isomorphism. The claims (ii) and (iii) now follow from proposition 5.3.1.

As $\mathbf{D}_K^{\dagger,s}(V)$ is independent of L, we can assume that $K, K' \subset L$ holds. Hence, $\mathbf{D}_K^{\dagger,s}(V) = \mathbf{D}_{K'}^{\dagger,s}(V)^{H_K/H_{K'}}$ and proposition 5.3.1 proves (iv).

The proof of [Liu15, Thm. 1.1.4(4)] also works in our situation and proves (v).

(vi) follows directly from proposition 5.4.4. ☐

Theorem 5.4.8. *Assume that we are in situation 5.4.1 and $s \geq s(V)$. Then $\mathbf{D}_K^{\dagger,s}(V)$ is a (φ, Γ_K)-module over $S \widehat{\otimes} \mathbf{B}_K^{\dagger,s}$. If $K = L$ holds it is even an étale (φ, Γ_K)-module.*

Proof. We essentially follow [Liu15, Prop. 1.1.5]. The $\mathcal{O}_S \widehat{\otimes} \mathbf{A}_K^{\dagger,s}$-module

$$\mathbf{D}_{\mathrm{int},K}^{\dagger,s}(T) := ((\mathcal{O}_S \widehat{\otimes} \mathbf{A}_L^{\dagger,s}) \otimes_{\mathcal{O}_S \widehat{\otimes} \mathbf{A}_L^{\dagger,s(V)}} \varphi^{n(V)}(\mathbf{D}_{L,n(V)}^{\dagger,(p-1)/p}(T)))^{H_K}$$

is an integral model of $\mathbf{D}_K^{\dagger,s}(V)$. We have

$$\begin{aligned}
&(\mathcal{O}_S \widehat{\otimes} \mathbf{A}_L^{\dagger,ps}) \otimes_{\varphi, \mathcal{O}_S \widehat{\otimes} \mathbf{A}_L^{\dagger,s}} \varphi(\mathbf{D}_{\mathrm{int},L}^{\dagger,s}(T)) \\
&\quad = (\mathcal{O}_S \widehat{\otimes} \mathbf{A}_L^{\dagger,ps}) \otimes_{\varphi, \mathcal{O}_S \widehat{\otimes} \mathbf{A}_L^{\dagger,s}} \varphi\left((\mathcal{O}_S \widehat{\otimes} \mathbf{A}_L^{\dagger,s}) \otimes_{\mathcal{O}_S \widehat{\otimes} \mathbf{A}_L^{\dagger,s(V)}} \varphi^{n(V)}(\mathbf{D}_{L,n(V)}^{\dagger,(p-1)/p}(T))\right)
\end{aligned}$$

$$
\begin{aligned}
&= (\mathcal{O}_S \widehat{\otimes} \mathbf{A}_L^{\dagger,ps}) \otimes_{\varphi, \mathcal{O}_S \widehat{\otimes} \mathbf{A}_L^{\dagger,s(V)}} \varphi^{n(V)+1}(\mathbf{D}_{L,n(V)}^{\dagger,(p-1)/p}(T)) \\
&= (\mathcal{O}_S \widehat{\otimes} \mathbf{A}_L^{\dagger,ps}) \otimes_{\mathcal{O}_S \widehat{\otimes} \mathbf{A}_L^{\dagger,ps(V)}} (\mathcal{O}_S \widehat{\otimes} \mathbf{A}_L^{\dagger,ps(V)}) \otimes_{\varphi, \mathcal{O}_S \widehat{\otimes} \mathbf{A}_L^{\dagger,s(V)}} \varphi^{n(V)+1}(\mathbf{D}_{L,n(V)}^{\dagger,(p-1)/p}(T)) \\
&= (\mathcal{O}_S \widehat{\otimes} \mathbf{A}_L^{\dagger,ps}) \otimes_{\mathcal{O}_S \widehat{\otimes} \mathbf{A}_L^{\dagger,ps(V)}} \varphi^{n(V)+1} \left((\mathcal{O}_S \widehat{\otimes} \mathbf{A}_{L,n(V)+1}^{\dagger,(p-1)/p}) \otimes_{\mathcal{O}_S \widehat{\otimes} \mathbf{A}_{L,n(V)}^{\dagger,(p-1)/p}} \mathbf{D}_{L,n(V)}^{\dagger,(p-1)/p}(T) \right) \\
&= (\mathcal{O}_S \widehat{\otimes} \mathbf{A}_L^{\dagger,ps}) \otimes_{\mathcal{O}_S \widehat{\otimes} \mathbf{A}_L^{\dagger,ps(V)}} \varphi^{n(V)+1}(\mathbf{D}_{L,n(V)+1}^{\dagger,(p-1)/p}(T)) \\
&= \mathbf{D}_{\mathrm{int},L}^{\dagger,ps}(T)
\end{aligned}
$$

where the last equality holds due to the proof of the independence of $n(V)$ in proposition 5.4.7(i) which also works integrally as it just uses corollary 5.4.5(i).

We deduce that $\mathbf{D}_K^{\dagger,s}(V)$ is an étale φ-module if $K = L$.

Now proposition 5.4.7(iv) yields

$$
\left(S \widehat{\otimes} \mathbf{B}_L^{\dagger,ps}\right) \otimes_{\varphi, \mathcal{O}_S \widehat{\otimes} \mathbf{A}_K^{\dagger,s}} \varphi(\mathbf{D}_{\mathrm{int},K}^{\dagger,s}(T)) = \left(S \widehat{\otimes} \mathbf{B}_L^{\dagger,ps}\right) \otimes_{\mathcal{O}_S \widehat{\otimes} \mathbf{A}_K^{\dagger,ps}} \mathbf{D}_{\mathrm{int},K}^{\dagger,ps}(T),
$$

taking H_K-invariants gives

$$
\left(S \widehat{\otimes} \mathbf{B}_K^{\dagger,ps}\right) \otimes_{\varphi, \mathcal{O}_S \widehat{\otimes} \mathbf{A}_K^{\dagger,s}} \varphi(\mathbf{D}_{\mathrm{int},K}^{\dagger,s}(T)) = \left(S \widehat{\otimes} \mathbf{B}_K^{\dagger,ps}\right) \otimes_{\mathcal{O}_S \widehat{\otimes} \mathbf{A}_K^{\dagger,ps}} \mathbf{D}_{\mathrm{int},K}^{\dagger,ps}(T)
$$

which proves that $\mathbf{D}_K^{\dagger,s}(V)$ is indeed a φ-module.

Moreover $\mathbf{D}_K^{\dagger,s}(V)$ is a (φ, Γ_K)-module since the actions of Γ_K and φ commute because the Galois action commutes with φ on $(S \widehat{\otimes} \tilde{\mathbf{B}}^{\dagger,s}) \otimes_S V$. □

Definition 5.4.9. Moreover, we define

$$
\tilde{\mathbf{D}}_K^{\dagger(,s)}(V) \coloneqq (S \widehat{\otimes} \tilde{\mathbf{B}}_K^{\dagger(,s)}) \otimes_{S \widehat{\otimes} \mathbf{B}_K^{\dagger(,s)}} \mathbf{D}_K^{\dagger(,s)}(V)
$$

and

$$
\mathbf{D}_{\mathrm{rig},K}^{\dagger(,s)}(V) \coloneqq (S \widehat{\otimes} \mathbf{B}_{\mathrm{rig},K}^{\dagger,(s)}) \otimes_{S \widehat{\otimes} \mathbf{B}_K^{\dagger(,s)}} \mathbf{D}_K^{\dagger(,s)}(V).
$$

Remark 5.4.10. $\tilde{\mathbf{D}}_K^{\dagger(,s)}(V)$ and $\mathbf{D}_{\mathrm{rig},K}^{\dagger(,s)}(V)$ are also (φ, Γ_K)-modules over $S \widehat{\otimes} \tilde{\mathbf{B}}_K^{\dagger(,s)}$ and $S \widehat{\otimes} \mathbf{B}_{\mathrm{rig},K}^{\dagger,(s)}$ respectively.

5.5 Properties of $\mathrm{D}_K^{\dagger,s}(V)$

Lemma 5.5.1. *Let L be a finite extension of K and assume that $(S, V, T, F/L)$ as well as $(S, \mathrm{Ind}_{\mathrm{Gal}_L}^{\mathrm{Gal}_K} V, \mathrm{Ind}_{\mathrm{Gal}_L}^{\mathrm{Gal}_K} T, F/K)$ fulfils hypothesis 5.4.1. Then there is an isomorphism of left $S \widehat{\otimes} \mathbf{B}_K^{\dagger(,s)}$-modules*

$$\mathrm{Ind}_L^K \mathbf{D}_L^{\dagger(,s)}(V) \xrightarrow{\sim} \mathbf{D}_K^{\dagger(,s)}(\mathrm{Ind}_{\mathrm{Gal}_L}^{\mathrm{Gal}_K} V)$$

for $s \gg 0$. Analogous statements hold for $\tilde{\mathbf{D}}_K^{\dagger(,s)}(-)$ and $\mathbf{D}_{\mathrm{rig},K}^{\dagger(,s)}(-)$.

Proof. Lemma 4.4.1 yields $\mathbf{D}_F^{\dagger,s}(\mathrm{Ind}_{\mathrm{Gal}_L}^{\mathrm{Gal}_K} V) = \mathrm{Ind}_{\mathrm{Gal}_L}^{\mathrm{Gal}_K} \mathbf{D}_F^{\dagger,s}(V)$, hence we are reduced to showing that

$$\left(\mathbb{Z}[\mathrm{Gal}_K] \otimes_{\mathbb{Z}[\mathrm{Gal}_L]} \mathbf{D}_F^{\dagger,s}(V)\right)^{H_K} \cong \mathbb{Z}[\Gamma_K] \otimes_{\mathbb{Z}[\Gamma_L]} \mathbf{D}_F^{\dagger,s}(V)^{H_L}.$$

Note that H_F is normal in Gal_K and $\mathbf{D}_F^{\dagger,s}(V)$ is H_F invariant, hence the above statement is equivalent to

$$\left(\mathbb{Z}[\mathrm{Gal}_K/H_F] \otimes_{\mathbb{Z}[\mathrm{Gal}_L/H_F]} \mathbf{D}_F^{\dagger,s}(V)\right)^{H_K/H_F} \cong \mathbb{Z}[\Gamma_K] \otimes_{\mathbb{Z}[\Gamma_L]} \mathbf{D}_F^{\dagger,s}(V)^{H_L/H_F}.$$

We note first that a general element of $\left(\mathbb{Z}[\mathrm{Gal}_K/H_F] \otimes_{\mathbb{Z}[\mathrm{Gal}_L/H_F]} \mathbf{D}_F^{\dagger,s}(V)\right)^{H_L/H_F}$ looks like

$$\sum_i \alpha_i \sigma_i \otimes d_i = \frac{1}{[H_L : H_F]} \sum_i \sum_{\tau \in H_L/H_F} \tau \alpha_i \sigma_i \otimes d_i = \frac{1}{[H_L : H_F]} \sum_i \alpha_i \sigma_i \otimes \left(\sum_{\tau \in H_L/H_F} \tau d_i \right)$$

where $\alpha_i \in \mathbb{Z}$, $\sigma_i \in \mathrm{Gal}_K/H_F$ and $d_i \in \mathbf{D}_F^{\dagger,s}(V)$. The first equality holds because the element is invariant under H_L/H_F and the second equality is based on the following observation: let $\sigma \in \mathrm{Gal}_K/H_F$ and let S be representatives of H_L/H_F. Then $\sigma^{-1}S\sigma$ is still a system of representatives of H_L/H_F since H_L is normal in Gal_K. Hence

$$\mathbb{Z}[\mathrm{Gal}_K/H_F] \otimes_{\mathbb{Z}[\mathrm{Gal}_L/H_F]} \mathbf{D}_F^{\dagger,s}(V)^{H_L/H_F} \to \left(\mathbb{Z}[\mathrm{Gal}_K/H_F] \otimes_{\mathbb{Z}[\mathrm{Gal}_L/H_F]} \mathbf{D}_F^{\dagger,s}(V)\right)^{H_L/H_F}$$

is an isomorphism and the first term is isomorphic to $\mathbb{Z}[\mathrm{Gal}_K/H_L] \otimes_{\mathbb{Z}[\Gamma_L]} \mathbf{D}_F^{\dagger,s}(V)^{H_L/H_F}$.

Consider the maps

$$\begin{aligned} \mathbb{Q}[\mathrm{Gal}_K/H_L]^{H_K/H_L} &\to \mathbb{Q}[\Gamma_K] \\ \sigma &\mapsto \overline{\sigma} \\ \frac{1}{[H_K : H_L]} \sum_{\tau \in H_K/H_L} \tau\sigma &\hookleftarrow \overline{\sigma} \end{aligned}$$

and since the maps are mutually inverse, the spaces are isomorphic.

The desired statement now follows. □

Lemma 5.5.2. *Let S and S' be orthonormalisable $\mathbb{Q}_p$-Banach algebras. Let $\mathcal{Y}$ be a continuous $\mathcal{O}_{S'}$-$\mathcal{O}_S$-bi-module which is finitely generated, projective as a topological left $\mathcal{O}_{S'}$-Banach-module and set $Y = S' \otimes_{\mathcal{O}_{S'}} \mathcal{Y}$. If $(S, V, T, L/K)$ fulfils hypothesis 5.4.1, then there is an isomorphism of left $S' \mathbin{\widehat{\otimes}} \mathbf{B}_K^{\dagger(,s)}$-modules*

$$Y \mathbin{\widehat{\otimes}}_S \mathbf{D}_K^{\dagger(,s)}(V) \xrightarrow{\sim} \mathbf{D}_K^{\dagger(,s)}(Y \otimes_S V).$$

Similar statements hold for $\tilde{\mathbf{D}}_K^{\dagger(,s)}(-)$ and $\mathbf{D}_{\mathrm{rig},K}^{\dagger(,s)}(-)$.

Proof. The statement is clear for $\mathbf{D}_L^{\dagger,s}(V)$ using lemma 4.4.2 and the fact that base change commutes with φ and the inversion of p. Furthermore, one has to use that $S \mathbin{\widehat{\otimes}} \mathbf{B}_L^{\dagger,s} \otimes_{S \mathbin{\widehat{\otimes}} \mathbf{B}_L^{\dagger,s(V)}} M$ is isomorphic to $\mathbf{B}_L^{\dagger,s} \mathbin{\widehat{\otimes}}_{\mathbf{B}_L^{\dagger,s(V)}} M$ (see corollary 2.2.6).

Hence, it suffices to show that

$$Y \mathbin{\widehat{\otimes}}_S \mathbf{D}_L^{\dagger,s}(V)^{H_K} \to \left(Y \mathbin{\widehat{\otimes}}_S \mathbf{D}_L^{\dagger,s}(V)\right)^{H_K}$$

is an isomorphism. By lemma 5.3.3 this is equivalent to showing that

$$\left(S' \mathbin{\widehat{\otimes}} \mathbf{B}_L^{\dagger,s}\right) \otimes_{S' \mathbin{\widehat{\otimes}} \mathbf{B}_K^{\dagger,s}} \left(Y \mathbin{\widehat{\otimes}}_S \mathbf{D}_L^{\dagger,s}(V)^{H_K}\right) \to Y \mathbin{\widehat{\otimes}}_S \mathbf{D}_L^{\dagger,s}(V)$$

is an isomorphism. Indeed:

$$\begin{aligned} &\left(S' \mathbin{\widehat{\otimes}} \mathbf{B}_L^{\dagger,s}\right) \otimes_{S' \mathbin{\widehat{\otimes}} \mathbf{B}_K^{\dagger,s}} \left(Y \mathbin{\widehat{\otimes}}_S \mathbf{D}_L^{\dagger,s}(V)^{H_K}\right) \\ &\xrightarrow{\sim} \mathbf{B}_L^{\dagger,s} \otimes_{\mathbf{B}_K^{\dagger,s}} \left(Y \mathbin{\widehat{\otimes}}_S \mathbf{D}_L^{\dagger,s}(V)^{H_K}\right) \\ &\xrightarrow{\sim} Y \mathbin{\widehat{\otimes}}_S \left(\mathbf{B}_L^{\dagger,s} \otimes_{\mathbf{B}_K^{\dagger,s}} \mathbf{D}_L^{\dagger,s}(V)^{H_K}\right) \end{aligned}$$

$$\xrightarrow{\sim} Y \,\widehat{\otimes}_S \left(\left(S \,\widehat{\otimes}\, \mathbf{B}_L^{\dagger,s} \right) \otimes_{S \widehat{\otimes} \mathbf{B}_K^{\dagger,s}} \mathbf{D}_L^{\dagger,s}(V)^{H_K} \right)$$
$$\xrightarrow{\sim} Y \,\widehat{\otimes}_S \mathbf{D}_L^{\dagger,s}(V)$$

where we used lemma 5.3.3 again to deduce the last isomorphism. □

Lemma 5.5.3. *Let S and S' be orthonormalisable $\mathbb{Q}_p$-Banach algebras. We assume that S is a* commutative *$\mathbb{Q}_p$-Banach algebra and that there is a continuous ring homomorphism $S \to S'$ which comes from an integral map $\mathcal{O}_S \to \mathcal{O}_{S'}$. Let $(S, V, T, L/K)$ and $(S', V', T', L/K)$ fulfil hypothesis 5.4.1. Then there is an isomorphism of left $S' \,\widehat{\otimes}\, \mathbf{B}_K^{\dagger(,s)}$-modules*

$$\mathbf{D}_K^{\dagger(,s)}(V_{S'}) \otimes_{S' \widehat{\otimes} \mathbf{B}_K^{\dagger(,s)}} \mathbf{D}_K^{\dagger(,s)}(V') \xrightarrow{\sim} \mathbf{D}_K^{\dagger(,s)}(V_{S'} \otimes_{S'} V').$$

Similar statements hold for $\tilde{\mathbf{D}}_K^{\dagger(,s)}(-)$ and $\mathbf{D}_{\mathrm{rig},K}^{\dagger(,s)}(-)$.

Proof. The statement is clear for $\mathbf{D}_L^{\dagger,s}$ using lemma 4.4.3 and the fact that the tensor product is compatible with the inversion of p, application of φ and extension of scalars.

Hence, it suffices to show that

$$\mathbf{D}_L^{\dagger,s}(V_{S'})^{H_K} \otimes_{S' \widehat{\otimes} \mathbf{B}_K^{\dagger,s}} \mathbf{D}_L^{\dagger,s}(V')^{H_K} \to \left(\mathbf{D}_L^{\dagger,s}(V_{S'}) \otimes_{S' \widehat{\otimes} \mathbf{B}_L^{\dagger,s}} \mathbf{D}_L^{\dagger,s}(V') \right)^{H_K}$$

is an isomorphism. By lemma 5.3.3 this is equivalent to showing that

$$\left(S' \,\widehat{\otimes}\, \mathbf{B}_L^{\dagger,s} \right) \otimes_{S' \widehat{\otimes} \mathbf{B}_K^{\dagger,s}} \left(\mathbf{D}_L^{\dagger,s}(V_{S'})^{H_K} \otimes_{S' \widehat{\otimes} \mathbf{B}_K^{\dagger,s}} \mathbf{D}_L^{\dagger,s}(V')^{H_K} \right) \to \mathbf{D}_L^{\dagger,s}(V_{S'}) \otimes_{S' \widehat{\otimes} \mathbf{B}_L^{\dagger,s}} \mathbf{D}_L^{\dagger,s}(V')$$

is an isomorphism. Indeed:

$$\left(S' \,\widehat{\otimes}\, \mathbf{B}_L^{\dagger,s} \right) \otimes_{S' \widehat{\otimes} \mathbf{B}_K^{\dagger,s}} \left(\mathbf{D}_L^{\dagger,s}(V_{S'})^{H_K} \otimes_{S' \widehat{\otimes} \mathbf{B}_K^{\dagger,s}} \mathbf{D}_L^{\dagger,s}(V')^{H_K} \right)$$
$$\xrightarrow{\sim} \left(\mathbf{D}_L^{\dagger,s}(V_{S'})^{H_K} \otimes_{S' \widehat{\otimes} \mathbf{B}_K^{\dagger,s}} \left(S' \,\widehat{\otimes}\, \mathbf{B}_L^{\dagger,s} \right) \right) \otimes_{S' \widehat{\otimes} \mathbf{B}_L^{\dagger,s}} \left(\left(S' \,\widehat{\otimes}\, \mathbf{B}_L^{\dagger,s} \right) \otimes_{S' \widehat{\otimes} \mathbf{B}_K^{\dagger,s}} \mathbf{D}_L^{\dagger,s}(V')^{H_K} \right)$$
$$\xrightarrow{\sim} \mathbf{D}_L^{\dagger,s}(V_{S'}) \otimes_{S' \widehat{\otimes} \mathbf{B}_L^{\dagger,s}} \mathbf{D}_L^{\dagger,s}(V')$$

where we used lemma 5.3.3 again to deduce the last isomorphism. □

5.6 Review of p-adic Hodge Theory for (φ, Γ_K)-Modules

In this section we will review p-adic Hodge theory as presented, for example, in [Nak13, §2.2] and [Pot13, §3.1]. *Hence we restrict ourselves to the (commutative) setting of affinoids.* However we note that many definitions and results generalise to our non-commutative setting.

According to [Pot13, §3.1] there exist Γ_K-equivariant maps $\iota_n : \mathbf{B}^{\dagger,p^n}_{\mathrm{rig},K} \to K_n[[t]]$ for $n \geq n(K)$ where $K_n := K(\zeta_{p^n})$. Furthermore the equality $\iota_{n+1} \circ \varphi = \iota_n$ holds.

Definition 5.6.1. For a (φ, Γ_K)-module D over $A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},K}$ where A is a (commutative) affinoid algebra with $A \widehat{\otimes} \mathbf{B}^{\dagger,s}_{\mathrm{rig},K}$-model D^s we define the following modules:

$$
\begin{aligned}
\mathbf{D}^{+}_{\mathrm{dif},K}(D) &:= \varprojlim \left(A \widehat{\otimes} K_n[[t]] \otimes_{A \widehat{\otimes} \mathbf{B}^{\dagger,s(n)}_{\mathrm{rig},K}} D^{s(n)} \right) && \text{[Nak13, pp. 12/13]} \\
\mathbf{D}_{\mathrm{dif},K}(D) &:= \varprojlim \left(A \widehat{\otimes} K_n((t)) \otimes_{A \widehat{\otimes} \mathbf{B}^{\dagger,s(n)}_{\mathrm{rig},K}} D^{s(n)} \right) && \text{[Nak13, pp. 12/13]} \\
\mathbf{D}^{(+)}_{\mathrm{dR},K}(D) &:= (\mathbf{D}^{(+)}_{\mathrm{dif},K}(D))^{\Gamma_K} && \text{[Nak13, p. 19]} \\
\mathbf{D}^{+}_{\mathrm{crys},K}(D) &:= D^{\Gamma_K} && \text{[Pot13, §3.1]} \\
\mathbf{D}_{\mathrm{crys},K}(D) &:= D[t^{-1}]^{\Gamma_K} && \text{[Nak13, p. 19].}
\end{aligned}
$$

The resulting objects land in the expected categories, e.g. $\mathbf{D}^{(+)}_{\mathrm{crys},K}(D)$ has an action of φ and the modules are filtered.

Definition 5.6.2. Let $A = L$ be a finite extension of $\mathbb{Q}_p$. A (φ, Γ_K)-module D over $A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},K}$ is called *crystalline* if

$$\mathrm{rk}_{A \otimes N_K} \mathbf{D}_{\mathrm{crys},K}(D) = \mathrm{rk}_{A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},K}} D$$

where N_K is the maximal unramified extension of $\mathbb{Q}_p$ in K.

D is called *deRham* if

$$\mathrm{rk}_{A \otimes K} \mathbf{D}_{\mathrm{dR},K}(D) = \mathrm{rk}_{A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},K}} D.$$

Next we define a tool which computes the f-part of the local cohomology.

Definition 5.6.3 ([Pot13, §3.1],[Nak13, p. 25])**.** Let

$$C^{\bullet}_{f}(K, D) := \mathrm{Fib}\left[\mathbf{D}_{\mathrm{crys},K}(D) \oplus \mathbf{D}^{+}_{\mathrm{dR},K}(D) \xrightarrow{(x,y) \to ((1-\varphi)(x), x-y)} \mathbf{D}_{\mathrm{crys},K}(D) \oplus \mathbf{D}_{\mathrm{dR},K}(D) \right]$$

(see definition 1.9.1 for the definition of the mapping fibre). Furthermore, define $\mathbf{R}\Gamma_f(K, D)$ to be the image of $C_f^\bullet(K, D)$ in the derived category and we denote the cohomology groups of $C_f^\bullet(K, D)$ by $H_f^i(K, D)$.

Remark 5.6.4. One can also define the e-part and the g-part of the local cohomology (see [Pot13, §3.1] and [Nak13, p. 25]).

Remark 5.6.5. Assuming that $K = \mathbb{Q}_p$ and $A = L$ is a finite extension of $\mathbb{Q}_p$, we can immediately deduce

$$\begin{aligned}
&\dim_L H_f^1(\mathbb{Q}_p, D) - \dim_L H_f^0(\mathbb{Q}_p, D) \\
&\quad = \left(\dim_L \mathbf{D}_{\mathrm{crys},\mathbb{Q}_p}(D) + \dim_L \mathbf{D}_{\mathrm{dR},\mathbb{Q}_p}(D)\right) - \left(\dim_L \mathbf{D}_{\mathrm{crys},\mathbb{Q}_p}(D) + \dim_L \mathbf{D}^+_{\mathrm{dR},\mathbb{Q}_p}(D)\right) \\
&\quad = \dim_L \mathbf{D}_{\mathrm{dR},\mathbb{Q}_p}(D) - \dim_L \mathbf{D}^+_{\mathrm{dR},\mathbb{Q}_p}(D)
\end{aligned}$$

from the definition.

Proposition 5.6.6 ([Nak13, Prop. 2.25(i)])**.** *Let $A = L$ be a finite extension of $\mathbb{Q}_p$. Then there are the following functorial isomorphisms:*

$$\begin{aligned}
\mathbf{D}^{(+)}_{\mathrm{dR},K}(V) &\xrightarrow{\sim} \mathbf{D}^{(+)}_{\mathrm{dR},K}(\mathbf{D}^\dagger_{\mathrm{rig},K}(V)), \\
\mathbf{D}^{(+)}_{\mathrm{crys},K}(V) &\xrightarrow{\sim} \mathbf{D}^{(+)}_{\mathrm{crys},K}(\mathbf{D}^\dagger_{\mathrm{rig},K}(V)), \\
C_f^\bullet(K, V) &\xrightarrow{\sim} C_f^\bullet(K, \mathbf{D}^\dagger_{\mathrm{rig},K}(V)).
\end{aligned}$$

5.7 Galois Cohomology

We are very close to [Pot13, §2.2/3].

Definition 5.7.1. In this section $\mathbf{B}_K^{(s)}$ denotes one of the following rings: $S \widehat{\otimes} \mathbf{B}_K^{\dagger,(s)}$, $S \widehat{\otimes} \mathbf{B}^{\dagger,(s)}_{\mathrm{rig},K}$ or $S \widehat{\otimes} \tilde{\mathbf{B}}_K^{\dagger,(s)}$ where S is an orthonormalisable $\mathbb{Q}_p$-Banach algebra.

Definition 5.7.2. For a (φ, Γ_K)-module over $\mathbf{B}_K^{(s)}$ we define the *Herr complex* to be

$$\begin{aligned}
\mathbf{R}\Gamma(K, D^{(s)}) &:= \mathbf{R}\Gamma_{\mathrm{cts}}\left(\Gamma_K, \mathrm{Fib}\left[D^{(s)} \xrightarrow{\varphi-1} D^{(ps)}\right]\right) \\
&\cong \mathrm{Fib}\left[\mathbf{R}\Gamma_{\mathrm{cts}}(\Gamma_K, D^{(s)}) \xrightarrow{\varphi-1} \mathbf{R}\Gamma_{\mathrm{cts}}(\Gamma_K, D^{(ps)})\right]
\end{aligned}$$

in $\mathbf{D}^b(S)$. We call $\mathbf{R}\Gamma(K, D^{(s)})$ the *Galois cohomology* of $D^{(s)}$.

Remark 5.7.3. As in the commutative case the Galois cohomology of $D^{(s)}$ can be represented by the complex

$$C^\bullet_{\varphi,\gamma_K}(D^{(s)}) = \left[(D^{(s)})^\Delta \xrightarrow{(\varphi-1,\gamma_K-1)} (D^{(ps)})^\Delta \oplus (D^{(s)})^\Delta \xrightarrow{(\gamma_K-1,1-\varphi)} (D^{(ps)})^\Delta\right]$$

concentrated in degrees $[0,2]$ where $\Delta \subset \Gamma_K$ is a finite group such that Γ_K/Δ is pro-cyclic, with topological generator γ_K. It is indeed a complex as φ and γ_K commute. Moreover $C^\bullet_{\varphi,\gamma_K}(D^{(s)})$ is independent of γ_K (see [KPX14, Def. 2.3.3]).

Lemma 5.7.4. *For a (φ, Γ_L)-module D^s over $\mathbf{B}_L^{(s)}$ the natural map*

$$\mathbf{R}\Gamma(L, D^{(s)}) \to \mathbf{R}\Gamma(K, \operatorname{Ind}_L^K D^{(s)})$$

is a quasi-isomorphism.

Proof. As in [KPX14, Lem. 2.3.5] it suffices to show that the natural map

$$\mathbf{R}\Gamma_{\mathrm{cts}}(\Gamma_L, D^{(s)}) \to \mathbf{R}\Gamma_{\mathrm{cts}}(\Gamma_K, \operatorname{Ind}_L^K D^{(s)})$$

is a quasi-isomorphism. Hence Shapiro's lemma for $\Gamma_L \subset \Gamma_K$ yields the statement. $\square$

Remark 5.7.5. We naturally have maps

$$\begin{aligned} \mathrm{res}_{L/K} : \quad & \mathbf{R}\Gamma(K, D^{(s)}) \to \mathbf{R}\Gamma(L, \operatorname{Res}_K^L D^{(s)}) \qquad \text{and} \\ \mathrm{cores}_{L/K} : \quad & \mathbf{R}\Gamma(L, \operatorname{Res}_K^L D^{(s)}) \to \mathbf{R}\Gamma(K, D^{(s)}) \end{aligned}$$

defined as in Pottharst's thesis. The composition $\mathrm{cores}_{L/K} \circ \mathrm{res}_{L/K}$ is as usual the multiplication-by-$[L : K]$ map on the cohomology groups.

Since $[L : K]$ is invertible, we might deduce that $H^i(K, D^{(s)})$ is canonically a direct summand of $H^i(L, \operatorname{Res}_K^L D^{(s)})$. Furthermore, this decomposition is respected by the maps induced by $D^{(s)} \to \tilde{D}^{(s)}$ and $D^{(s)} \to D^{(s)}_{\mathrm{rig}}$.

Due to technical difficulties we cannot adapt the base change proof presented by Kedlaya-Pottharst-Xiao, the best we can currently show in the non-commutative case is the following:

Proposition 5.7.6. *Assume that S is noetherian. Let Y be a S'-S-bi-module such that Y over S' is finitely generated and projective and the right action of S on Y is continuous and commutes with the left action of S'. We assume furthermore that Y is* finitely generated *as an S-module. Let $D^{(s)}$ be a (φ, Γ_K)-module over $S \mathbin{\widehat{\otimes}} \mathbf{B}^{\dagger,(s)}_{\mathrm{rig},K}$. Then*

$$Y \otimes^L_S \mathbf{R}\Gamma(K, D^{(s)}) \xrightarrow{\sim} \mathbf{R}\Gamma(K, Y \mathbin{\widehat{\otimes}}_S D^{(s)})$$

in the derived category.

Proof. D^s and D^{ps} are flat over S by lemma 5.1.3. Hence the statement reduces to showing that

$$C^\bullet_{\varphi,\gamma_K}(Y \otimes_S D^{(s)}) \to C^\bullet_{\varphi,\gamma_K}(Y \mathbin{\widehat{\otimes}}_S D^{(s)})$$

is a quasi-isomorphism. Since Y is finitely presented over S we have that $Y \otimes_S D^{(s)}$ is complete, i.e. $Y \otimes_S D^{(s)} = Y \mathbin{\widehat{\otimes}}_S D^{(s)}$. □

Also due to technical difficulties we are not able to generalise Kedlaya-Pottharst-Xiao's proof of perfectness of the complexes $\mathbf{R}\Gamma(K, D)$. However in the relevant situations we can use the following strategy to deduce perfectness: for a (φ, Γ_K)-module D over $S \mathbin{\widehat{\otimes}} \mathbf{B}^{\dagger}_{\mathrm{rig},K}$ the complex $\mathbf{R}\Gamma(K, D)$ is perfect if and only if $\mathbf{R}\Gamma(K, tD)$ is perfect. We follow the strategy in [KPX14, §3.2] and the proofs are the same. However, we still include most of the proofs for the convenience of the reader.

In [KPX14, Not. 3.2.2] they define maps $\iota_n : \mathcal{R}^{[r,s]} \to \mathbb{Q}_{p,n}$ such that $\iota_{n+1} \circ \varphi = (\mathbb{Q}_{p,n} \hookrightarrow \mathbb{Q}_{p,n+1}) \circ \iota_n$ if $r \le p^{1-n} \le s$. Since there is the isomorphism $\mathbf{B}^{\dagger,s}_{\mathrm{rig},K} \cong \mathcal{R}^{c_{\mathbb{Q}_p}/s}$, there are induced maps $\iota_n : S \mathbin{\widehat{\otimes}} \mathbf{B}^{\dagger,s}_{\mathrm{rig},K} \to S \otimes \mathbb{Q}_{p,n}$ if $s \le c_{\mathbb{Q}_p} p^{n-1}$. Define $D^s_n := (S \otimes \mathbb{Q}_{p,n}) \otimes_{S \mathbin{\widehat{\otimes}} \mathbf{B}^{\dagger,s}_{\mathrm{rig},\mathbb{Q}_p}} D^s$ assuming $s \le c_{\mathbb{Q}_p} p^{n-1}$. Then we have the following version of [KPX14, Lem. 3.2.3]:

Lemma 5.7.7. *Assume that S is an orthonormalisable $\mathbb{Q}_p$-Banach algebra. Let n_0 be the smallest integer such that $\log_p(s_0/c_{\mathbb{Q}_p}) \le n_0$. Then the following holds:*

(i) $D^{s_0}/t \cong \prod_{n \ge n_0} D^{s_0}_n$,

(ii) $1 \otimes \varphi^{n'-n}$ *induces an isomorphism*

$$\mathbb{Q}_{p,n'} \otimes_{\mathbb{Q}_{p,n}} D^{s_0}_n \cong D^{s_0}_{n'}$$

as $S[\Gamma_{\mathbb{Q}_p}]$-modules for $n' \ge n \ge n_0$,

(iii) the map $\varphi : D^{s_0}/t \to D^{ps_0}/t$ induces via the isomorphism (i) the map $\prod_{n \ge n_0} D^{s_0}_n \to \prod_{n \ge n_0+1} D^{ps_0}_n$ given by $(x_n)_n \mapsto (x_{n-1})_n$.

Proof. We have to use some notation introduced in proposition 5.2.13 to prove (i). According to the proof of [KPX14, Lem. 3.2.3(i)], we have

$$\mathcal{R}^{[r,c_{\mathbb{Q}_p}/s_0]}/t \xrightarrow{\sim} \oplus_{n_0 \le n \le -\log_p r} \mathbb{Q}_{p,n}.$$

Since $S \widehat{\otimes} -$ is exact and $- \otimes_{S \widehat{\otimes} \mathbf{B}^{\dagger,s_0}_{\mathrm{rig},\mathbb{Q}_p}} D^{s_0}$ is right exact, we also find

$$\left(S \widehat{\otimes} \mathcal{R}^{[r,c_{\mathbb{Q}_p}/s_0]}/t\right) \otimes_{S \widehat{\otimes} \mathbf{B}^{\dagger,s_0}_{\mathrm{rig},\mathbb{Q}_p}} D^{s_0} \xrightarrow{\sim} \oplus_{n_0 \le n \le -\log_p r} D^{s_0}_n.$$

(i) now follows because the projective limit of inverse systems with transition maps which have dense image commutes with tensoring with finitely presented modules (see the proof of corollary 3.7.12).

Regarding (ii), using $\iota_{n+1} \circ \varphi = (K_n \hookrightarrow K_{n+1}) \circ \iota_n$ we have

$$\begin{aligned}
\mathbb{Q}_{p,n+1} \otimes_{\mathbb{Q}_{p,n}} D^{s_0}_n &= \mathbb{Q}_{p,n+1} \otimes_{\mathbb{Q}_{p,n}} (S \widehat{\otimes} \mathbb{Q}_{p,n}) \otimes_{i_n, S \widehat{\otimes} \mathbf{B}^{\dagger,s_0}_{\mathrm{rig},\mathbb{Q}_p}} D^{s_0} \\
&= (S \widehat{\otimes} \mathbb{Q}_{p,n+1}) \otimes_{i_{n+1}\circ\varphi, S \widehat{\otimes} \mathbf{B}^{\dagger,s_0}_{\mathrm{rig},\mathbb{Q}_p}} D^{s_0} \\
&= (S \widehat{\otimes} \mathbb{Q}_{p,n+1}) \otimes_{i_{n+1}, S \widehat{\otimes} \mathbf{B}^{\dagger,ps_0}_{\mathrm{rig},\mathbb{Q}_p}} S \widehat{\otimes} \mathbf{B}^{\dagger,ps_0}_{\mathrm{rig},\mathbb{Q}_p} \otimes_{\varphi, S \widehat{\otimes} \mathbf{B}^{\dagger,s_0}_{\mathrm{rig},\mathbb{Q}_p}} D^{s_0} \\
&\cong (S \widehat{\otimes} \mathbb{Q}_{p,n+1}) \otimes_{i_{n+1}, S \widehat{\otimes} \mathbf{B}^{\dagger,ps_0}_{\mathrm{rig},\mathbb{Q}_p}} D^{ps_0} \\
&= D^{ps_0}_{n+1} \\
&= D^{s_0}_{n+1}
\end{aligned}$$

where we also used that D^{s_0} is a φ-module, i.e. the map

$$1 \otimes \varphi : \quad S \widehat{\otimes} \mathbf{B}^{\dagger,ps_0}_{\mathrm{rig},\mathbb{Q}_p} \otimes_{\varphi, S \widehat{\otimes} \mathbf{B}^{\dagger,s_0}_{\mathrm{rig},\mathbb{Q}_p}} D^{s_0} \to D^{ps_0}$$

is an isomorphism.

(iii) follows from $\iota_{n+1} \circ \varphi = (\mathbb{Q}_{p,n} \hookrightarrow \mathbb{Q}_{p,n+1}) \circ \iota_n$. □

Proposition 5.7.8. *The complex $\mathbf{R}\Gamma(\mathbb{Q}_p, D/t)$ is perfect if one uses the usual Herr complex.*

Proof. We assume that D comes from D^{s_0}, then lemma 5.7.7(i) implies $D^{s_0}/t \cong \prod_{n \ge n_0} D^{s_0}_n$.

We will first prove that $1 - \varphi : D^{s_0}/t \to D^{ps_0}/t$ is surjective by an explicit construction. Let $(x_n)_n \in \prod_{n \ge n_0+1} D^{ps_0}_n \cong D^{ps_0}/t$ be some element. Set $y = (y_n)_n$ with

$y_n = \sum_{m=n_0+1}^{n} x_m$. Then

$$(1-\varphi)(y)_n = y_n - y_{n-1} = \sum_{m=n_0+1}^{n} x_m - \sum_{m=n_0+1}^{n-1} x_m = x_n.$$

Furthermore,

$$(D^s/t)^{\varphi=1} \cong \left(\prod_{n \geq n_0(s)} D_n^s\right)^{\varphi=1} \xrightarrow{\sim} D^s_{n_0(s)}$$

and if one takes lemma 5.7.7(iii) into account, the last map is the projection map. Hence, $(D/t)^{\varphi=1} = \varinjlim_s (D^s/t)^{\varphi=1}$ is isomorphic to $\varinjlim_s D^s_{n_0(s)} = \varinjlim_n D_n^{s_0}$. Hence, $\mathbf{R}\Gamma(\mathbb{Q}_p, D/t)$ is isomorphic to the complex

$$\mathbf{R}\Gamma(\Gamma, \varinjlim_n D_n^{s_0}) = [(\varinjlim_n D_n^{s_0})^{\Delta} \xrightarrow{\gamma_K - 1} (\varinjlim_n D_n^{s_0})^{\Delta}].$$

To finish the proof we want to show that $[(\varinjlim_n D_n^{s_0})^{\Delta} \xrightarrow{\gamma_K - 1} (\varinjlim_n D_n^{s_0})^{\Delta}]$ is quasi-isomorphic to $[(D_{n_1}^{s_0})^{\Delta} \xrightarrow{\gamma_K - 1} (D_{n_1}^{s_0})^{\Delta}]$ for some $n_1 \geq n_0$. This is indeed a perfect complex as D is projective, hence $D_n^{s_0}$ is also projective for all n. It suffices to show that $\gamma_K - 1$ acts bijectively on D_n/D_{n_1} for $n \geq n_1$. Note that 5.7.7(ii) implies $D_n^{s_0} \cong \mathbb{Q}_{p,n} \otimes_{\mathbb{Q}_{p,n_1}} D_{n_1}^{s_0}$ as an $S[\Gamma_{\mathbb{Q}_p}]$-module for $n \geq n_1$.

According to [Liu08, Lem. 3.6(ii)] we have $\mathbb{Q}_{p,n} \cong \oplus_{\rho, N(\rho) \leq n} \mathbb{Q}_p \cdot G(\rho)$ as $\Gamma_{\mathbb{Q}_p}$-modules where ρ is an irreducible finite order $\mathbb{Q}_p$-valued character of $\Gamma_{\mathbb{Q}_p}$ with conductor $p^{N(\rho)}$ bounded by n and $G(\rho)$ is a Gauß sum as defined in [Liu08, p. 20]. Hence we are reduced to finding an n_1 such that on $D^{s_0}(\rho) = (S \otimes \mathbb{Q}_p \cdot G(\rho)) \otimes_{S \widehat{\otimes} \mathbf{B}^{\dagger,s}_{\mathrm{rig},\mathbb{Q}_p}} D^{s_0}$ the map $\gamma_{K,\rho} - 1$ is bijective for $N(\rho) \geq n_1$. We note that the $\gamma_{K,\rho} = \rho(\gamma_K) \cdot \gamma_K$ holds.

Now we can complete the proof along the lines of [KPX14, Prop. 3.2.4]. □

Corollary 5.7.9. *$\mathbf{R}\Gamma(K, D)$ is perfect if and only if $\mathbf{R}\Gamma(K, tD)$ is perfect.*

Proof. By lemma 5.7.4 $\mathbf{R}\Gamma(K, -)$ is perfect if and only if $\mathbf{R}\Gamma(\mathbb{Q}_p, \mathrm{Ind}_K^{\mathbb{Q}_p} -)$ is perfect. Hence the previous proposition implies the result. □

5.8 Galois Cohomology of Galois Representations

In this section we want to compare the Galois cohomology of p-adic representations and the Galois cohomology of its associated (φ, Γ_K)-module. We follow [Pot13, §2.2] very closely where Pottharst did the same using the virtually the same proofs in the commutative situation.

Proposition 5.8.1. *The natural map*

$$\mathbf{R}\Gamma(K, \mathbf{D}_K^\dagger(V)) \to \mathbf{R}\Gamma(K, \tilde{\mathbf{D}}_K^\dagger(V))$$

is an isomorphism in $\mathbf{D}^b(S)$.

Proof. Let L be a finite Galois extension of K such that Gal_L acts trivially on $T/12pT$. It suffices to show the claim for L instead of K as the direct sum decomposition of the cohomology gets respected by the morphisms (see remark 5.7.5).

It suffices to show that

$$\mathbf{R}\Gamma_{\mathrm{cts}}(\Gamma_L, \mathbf{D}_L^\dagger(V)) \to \mathbf{R}\Gamma_{\mathrm{cts}}(\Gamma_L, \tilde{\mathbf{D}}_L^\dagger(V))$$

is an isomorphism in the derived category (see definition 5.7.2).

By lemma 4.4.4, the $S \mathbin{\widehat{\otimes}} \tilde{\mathbf{B}}_L^\dagger$-module

$$\tilde{\mathbf{D}}_L^\dagger(V) = (S \mathbin{\widehat{\otimes}} \tilde{\mathbf{B}}_L^\dagger) \otimes_{S \mathbin{\widehat{\otimes}} \mathbf{B}_L^{\dagger, s(V)}} \varphi^{n(V)}(\mathbf{D}_{L,n(V)}^{\dagger,(p-1)/p}(T)[\tfrac{1}{p}])$$

decomposes into $\varphi^{n(V)}(\mathbf{D}_{L,n(V)}^{\dagger,(p-1)/p}(T))[1/p] \oplus X(T)[1/p]$ and moreover $\gamma_L - 1$ is invertible on $X(T)[1/p]$. Hence, $\mathbf{R}\Gamma_{\mathrm{cts}}(\Gamma_L, X(T)[1/p]) \cong 0$ and we deduce the isomorphism. □

Proposition 5.8.2. *The natural map*

$$\mathbf{R}\Gamma(K, \mathbf{D}_K^{\dagger,(s)}(V)) \to \mathbf{R}\Gamma(K, \mathbf{D}_{\mathrm{rig},K}^{\dagger,(s)}(V))$$

is an isomorphism in $\mathbf{D}^b(A)$ *for a* $\mathbb{Q}_p$*-nc-affinoid algebra* A.

Proof. Let L be a finite Galois extension of K such that Gal_L acts trivially on $T/12pT$. It suffices to show the claim for L instead of K as the direct sum decomposition of the cohomology gets respected by the morphisms (see remark 5.7.5).

$\mathbf{D}_L^{\dagger,(s)}(V)$ is étale by theorem 5.4.8, hence corollary 5.2.14 implies that

$$[\mathbf{D}_L^{\dagger,s}(V) \xrightarrow{1-\varphi} \mathbf{D}_L^{\dagger,ps}(V)] \to [\mathbf{D}_{\mathrm{rig},L}^{\dagger,s}(V) \xrightarrow{1-\varphi} \mathbf{D}_{\mathrm{rig},L}^{\dagger,ps}(V)]$$

is a quasi-isomorphism. The statement without s follows by taking $\varinjlim_s$ and noting that this functor is exact. □

We are now able to prove the main theorem of this section, the exact analogue of [Pot13, Thm. 2.8]:

Theorem 5.8.3. *For a* Gal_K*-representation* V *over an nc-affinoid algebra* A *there is a natural isomorphism*

$$\mathbf{R}\Gamma_{\mathrm{cts}}(K, V) \xrightarrow{\sim} \mathbf{R}\Gamma(K, \mathbf{D}^{\dagger}_{\mathrm{rig},K}(V))$$

in $\mathbf{D}^b(A)$.

Proof. We note that by propositions 5.8.1 and 5.8.2 we only have to show the existence of a natural isomorphism

$$\mathbf{R}\Gamma_{\mathrm{cts}}(K, V) \xrightarrow{\sim} \mathbf{R}\Gamma(K, \tilde{\mathbf{D}}^{\dagger}_{K}(V)).$$

Since the proof of [Pot13, Lem. 2.9] just uses the fact that A is orthonormalisable, the analogue for nc-affinoid still holds i.e.

$$A \to \mathrm{Cone}\left[A \mathbin{\widehat{\otimes}} \tilde{\mathbf{B}}^{\dagger,s} \xrightarrow{1-\varphi} A \mathbin{\widehat{\otimes}} \tilde{\mathbf{B}}^{\dagger,ps}\right][-1] \quad \text{and}$$
$$A \mathbin{\widehat{\otimes}} \tilde{\mathbf{B}}^{\dagger,s}_{K} \to \mathbf{R}\Gamma_{\mathrm{cts}}(H_K, A \mathbin{\widehat{\otimes}} \tilde{\mathbf{B}}^{\dagger,s})$$

are quasi-isomorphisms. Hence, tensoring the first quasi-isomorphism with V and using proposition 5.4.7(iii) we get that

$$V \to \mathrm{Cone}\left[(A \mathbin{\widehat{\otimes}} \tilde{\mathbf{B}}^{\dagger,s}) \otimes_{A \widehat{\otimes} \tilde{\mathbf{B}}^{\dagger,s}_{K}} \tilde{\mathbf{D}}^{\dagger,s}_{K}(V) \xrightarrow{1-\varphi} (A \mathbin{\widehat{\otimes}} \tilde{\mathbf{B}}^{\dagger,ps}) \otimes_{A \widehat{\otimes} \tilde{\mathbf{B}}^{\dagger,ps}_{K}} \tilde{\mathbf{D}}^{\dagger,ps}_{K}(V)\right][-1]$$

is a quasi-isomorphism. Applying $\mathbf{R}\Gamma_{\mathrm{cts}}(H_K, -)$ and noting that

$$\mathbf{R}\Gamma_{\mathrm{cts}}\left(H_K, (A \mathbin{\widehat{\otimes}} \tilde{\mathbf{B}}^{\dagger,s}) \otimes_{A \widehat{\otimes} \tilde{\mathbf{B}}^{\dagger,s}_{K}} \tilde{\mathbf{D}}^{\dagger,s}_{K}(V)\right) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cts}}\left(H_K, A \mathbin{\widehat{\otimes}} \tilde{\mathbf{B}}^{\dagger,s}\right) \otimes_{A \widehat{\otimes} \tilde{\mathbf{B}}^{\dagger,s}_{K}} \tilde{\mathbf{D}}^{\dagger,s}_{K}(V)$$

is a quasi-isomorphism together with the second quasi-isomorphism of [Pot13, Lem. 2.9] yields the quasi-isomorphism

$$\mathbf{R}\Gamma_{\mathrm{cts}}(H_K, V) \to \mathrm{Cone}\left[\tilde{\mathbf{D}}^{\dagger,s}_{K}(V) \xrightarrow{1-\varphi} \tilde{\mathbf{D}}^{\dagger,ps}_{K}(V)\right][-1].$$

Applying $\mathbf{R}\Gamma_{\mathrm{cts}}(\Gamma_K, -)$ and noting that $\mathbf{R}\Gamma_{\mathrm{cts}}(\mathrm{Gal}_K, -) = \mathbf{R}\Gamma_{\mathrm{cts}}(\Gamma_K, -) \circ \mathbf{R}\Gamma_{\mathrm{cts}}(H_K, -)$ holds in the derived category proves the desired statement. □

5.9 Galois Cohomology and Duality

We follow [KPX14, §2.3] and we assume in this section that all (φ, Γ_K)-modules are over $A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},K}$.

Definition 5.9.1. We have $A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},K}(1) = \mathbf{D}^\dagger_{\mathrm{rig},K}(A(1))$, hence

$$\mathbf{R}\Gamma_{\mathrm{cts}}(K, A(1)) \xrightarrow{\sim} \mathbf{R}\Gamma(K, A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},K}(1))$$

and since A is flat over $\mathbb{Q}_p$ we find

$$A \xrightarrow{\sim} H^2_{\mathrm{cts}}(K, A(1)) \xrightarrow{\sim} H^2(K, A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},K}(1))$$

and we call the composition *Tate isomorphism* Ta_K.

Definition 5.9.2. The construction in definition [KPX14, Def. 2.3.10] also yields a *cup product*

$$\cup_{\varphi,\gamma_K} : C^\bullet_{\varphi,\gamma_K}(D_1) \otimes_A C^\bullet_{\varphi,\gamma_K}(D_2) \to C^\bullet_{\varphi,\gamma_K}(D_1 \otimes_{A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},K}} D_2)$$

where D_1 is a right (φ, Γ_K)-module over $A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},K}$ and D_2 is a left (φ, Γ_K)-module over $A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},K}$. The dual of a left module D is the right (φ, Γ_K)-module

$$D^* := \mathrm{Hom}_{A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},K}}(D, A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},K})$$

and the definition of the actions can be found in [Rie13, §2.6.1]. We also define $D^D := D^*(1)$.

Definition 5.9.3. Let $ev : D^D \otimes_{A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},K}} D \to A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},K}(1)$ be the evaluation morphism. Then the composition of

$$\begin{aligned} C^\bullet_{\varphi,\gamma_K}(D^D) \otimes_A C^\bullet_{\varphi,\gamma_K}(D) &\xrightarrow{\cup_{\varphi,\gamma_K}} C^\bullet_{\varphi,\gamma_K}(D^D \otimes_{A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},K}} D) \xrightarrow{ev} C^\bullet_{\varphi,\gamma_K}(A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},K}(1)) \\ &\xrightarrow{\mathrm{proj}} H^2(K, A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},K}(1)) \xrightarrow{Ta_K} A[-2] \end{aligned}$$

is called the *Tate pairing.* The pairing induces a morphism

$$\Psi(\mathbb{Q}_p, D) : \mathbf{R}\Gamma(K, D) \to \mathbf{R}\mathrm{Hom}_A(\mathbf{R}\Gamma(K, D^D), A)[-2].$$

Remark 5.9.4. We are unable to prove that the Tate pairing is an isomorphism. Kedlaya-Pottharst-Xiao's proof in the commutative situation does not seem to generalise to our setting in a straightforward manner.

5.10 (φ, Γ_K)-Modules over Fréchet-Stein Algebras

Definition 5.10.1. Let A_∞ be a K-Fréchet-Stein algebra. We say that the projective system $D = (D_n)_n$ is a *(left)* (φ, Γ_K)*-module over* $(A_n \mathbin{\widehat{\otimes}} \mathbf{B}_K)_n$ if every D_n is a (left) (φ, Γ_K)-module over $A_n \mathbin{\widehat{\otimes}} \mathbf{B}_K$ and the induced morphisms

$$A_n \mathbin{\widehat{\otimes}}_{A_{n+1}} D_{n+1} \to D_n$$

are all isomorphisms of (φ, Γ_K)-modules.

We might abuse the notation by writing $A_\infty \mathbin{\widehat{\otimes}} \mathbf{B}_K$ instead of $(A_n \mathbin{\widehat{\otimes}} \mathbf{B}_K)_n$

Remark 5.10.2. By taking first the projective limit and then completing the result we can get actual modules over $A_\infty \mathbin{\widehat{\otimes}} \mathbf{B}_K$. This would however just complicate the arguments below without any benefit for us.

As expected we can associate a (φ, Γ_K)-module to a coadmissable module:

Definition 5.10.3. Let V_∞ be a coadmissable A_∞-module such that all $V_n \coloneqq A_n \otimes_{A_\infty} V$ fulfil hypothesis 5.4.1. Then we can define

$$\mathbf{D}^\dagger_{\mathrm{rig},K}(V_\infty) \coloneqq (\mathbf{D}^\dagger_{\mathrm{rig},K}(V_n))_n.$$

The projective system $\mathbf{D}^\dagger_{\mathrm{rig},K}(V_\infty)$ is indeed a (φ, Γ_K)-module over $(A_n \mathbin{\widehat{\otimes}} \mathbf{B}^\dagger_{\mathrm{rig},K})_n$ due to lemma 5.5.2.

We will now review the standard properties of $\mathbf{D}^\dagger_{\mathrm{rig},K}(V_\infty)$ along the lines of §5.5.

Remark 5.10.4. As in lemma 5.5.1 we find

$$\mathrm{Ind}_L^K \mathbf{D}^\dagger_{\mathrm{rig},L}(V_\infty) \xrightarrow{\sim} \mathbf{D}^\dagger_{\mathrm{rig},K}(\mathrm{Ind}_{\mathrm{Gal}_L}^{\mathrm{Gal}_K} V_\infty)$$

since it is true for every n.

Remark 5.10.5. Assume the same situation as in lemma 3.7.9, and additionally require the Banach algebras A_n and B_n to be orthonormalisable. Then

$$\begin{aligned} Y_\infty \mathbin{\widehat{\otimes}}_{A_\infty} \mathbf{D}^\dagger_{\mathrm{rig},K}(V_\infty) &\coloneqq (Y_n \mathbin{\widehat{\otimes}}_{A_{a(n)}} \mathbf{D}^\dagger_{\mathrm{rig},K}(V_{a(n)}))_n \\ &\xrightarrow{\sim} (\mathbf{D}^\dagger_{\mathrm{rig},K}(Y_n \otimes_{A_{a(n)}} V_{a(n)}))_n = \mathbf{D}^\dagger_{\mathrm{rig},K}(Y_\infty \otimes_{A_\infty} V_\infty) \end{aligned}$$

due to lemma 5.5.2.

Remark 5.10.6. Let A_∞ and B_∞ be two two-sided Fréchet-Stein algebras where A_∞ is *commutative*. Furthermore let $A_\infty \to B_\infty$ be a morphism where A_n factors through $B_{a(n)}$. We also assume that this morphism has an integral model. Then

$$\mathbf{D}^\dagger_{\mathrm{rig},K}((V_\infty)_{B_\infty}) \otimes_{B_\infty \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},K}} \mathbf{D}^\dagger_{\mathrm{rig},K}(V'_\infty) \xrightarrow{\sim} \mathbf{D}^\dagger_{\mathrm{rig},K}((V_\infty)_{B_\infty} \otimes_{B_\infty} V'_\infty).$$

where V_∞ is an A_∞-module and V'_∞ is a B_∞-module.

Definition 5.10.7. For a (φ, Γ_K)-module $D = (D_n)_n$ over $(A_n \widehat{\otimes} \mathbf{B}_K)_n$ we define

$$\mathbf{R}\Gamma(K, D) \coloneqq [C^\bullet_{\varphi,\gamma_K}(D_n)]_n \in \mathbf{D}_{\mathrm{sh}}(A_\infty).$$

Assuming $\mathbf{R}\Gamma(K, D_n) \in \mathbf{D}_{\mathrm{perf}}(A_n)$ for all n, then by definition $\mathbf{R}\Gamma(K, D) \in \mathbf{D}_{\mathrm{sh,perf}}(A_\infty)$.

Remark 5.10.8. Assume the same situation as in lemma 3.7.9, and additionally require the Banach algebras A_n and B_n to be orthonormalisable. Furthermore we assume that Y_n is *finitely generated* over $A_{a(n)}$. Then the analogue of proposition 5.7.6 holds:

$$\begin{aligned} Y_\infty \otimes^L_{A_\infty} \mathbf{R}\Gamma(K, D) &\coloneqq \left[Y_n \otimes^L_{A_{a(n)}} C^\bullet_{\varphi,\gamma_K}(D_n) \right]_n \\ &\xrightarrow{\sim} \mathbf{R}\Gamma(K, Y_\infty \widehat{\otimes}_{A_\infty} D) \coloneqq \mathbf{R}\Gamma(K, (Y_n \widehat{\otimes}_{A_{a(n)}} D_{a(n)})_n) \end{aligned}$$

in the derived category $\mathbf{D}_{\mathrm{sh}}(B_\infty)$ because it holds for every n.

Remark 5.10.9. As in theorem 5.8.3 there is a natural isomorphism

$$\mathbf{R}\Gamma(K, V_\infty) \xrightarrow{\sim} \mathbf{R}\Gamma(K, \mathbf{D}^\dagger_{\mathrm{rig},K}(V_\infty))$$

in the derived category since it holds on every level.

6 The Local Epsilon Conjecture for $(\varphi, \Gamma_{\mathbb{Q}_p})$-modules

Recall the notation from section 1.5 and section 2.6 regarding determinants.

Similar to Fukaya-Kato's local ϵ-isomorphism conjecture [FK06, §3], which states that there should exist canonical and strongly functorial isomorphisms

$$(\Lambda \mathbin{\widehat{\otimes}} \mathbb{Z}_p^{\mathrm{ur}}) \otimes_\Lambda \mathbf{1}_\Lambda \longrightarrow (\Lambda \mathbin{\widehat{\otimes}} \mathbb{Z}_p^{\mathrm{ur}}) \otimes_\Lambda \left(\mathbf{d}_\Lambda \, \mathbf{R}\Gamma_{\mathrm{cts}}(\mathbb{Q}_p, T) \cdot \mathbf{d}_\Lambda \, T \right)$$

for every $\mathbb{Q}_p$-representation, we would like to conjecture the same for $(\varphi, \Gamma_{\mathbb{Q}_p})$-modules. We note that in Fukaya-Kato's case they could just use $\mathbf{d}_\Lambda \, T$ as the 'target' but in our case we need to associate an object in the determinant category over A to a $(\varphi, \Gamma_{\mathbb{Q}_p})$-module. In the commutative case this was accomplished by Nakamura in [Nak13] using the Knudsen-Mumford determinant category, in our more general non-commutative case we can only construct the desired object in Deligne's determinant category for certain special cases. We conjecture the existence in a larger generality.

We then go on and conjecture the existence of ϵ-isomorphisms for a non-commutative (φ, Γ_K)-modules which generalises Fukaya-Kato's non-commutative conjecture for local Galois representations and Nakamura's commutative ϵ-conjecture for (φ, Γ_K)-modules.

6.1 The Primitive Determinants

From now on we assume the following for the remainder of this section:

Hypothesis 6.1.1. (i) We cover two cases:

(A) A is a $\mathbb{Q}_p$-nc-affinoid algebra and

(FS) $A = A_\infty = \varprojlim A_n$ is a two-sided Fréchet-Stein algebra.

(ii) D is a $(\varphi, \Gamma_{\mathbb{Q}_p})$-module over $A \mathbin{\widehat{\otimes}} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}$.

Definition 6.1.2. Assuming hypothesis 6.1.1, a *primitive determinant of D over A* is an element $\Psi'_A(D)$ of the universal determinant category $\mathbf{Det}(A)$ in case (A) or $\mathbf{Det}^{\mathrm{sh}}(A_\infty)$ in case (FS) together with

(i) an isomorphism

$$c_A(D) : A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p} \otimes_A \Psi'_A(D) \xrightarrow{\sim} \mathbf{d}_{A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}} D$$

in $\mathbf{Det}^{(\mathrm{sh})}(A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p})$ which we define in case (FS) as $\mathrm{QCohSh}(\mathbf{Det}(A_n \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p})_n)$,

(ii) an operator $\varphi \in \mathrm{Aut}_A(\Psi'_A(D))$ such that the following diagram commutes:

$$\begin{tikzcd}
A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p} \otimes_A \Psi'_A(D) \arrow[r, "c_A(D)"] \arrow[d, "1\otimes\varphi"] & A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p} \otimes_{\varphi, A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}} \mathbf{d}_{A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}} D \arrow[d, "\varphi"] \\
A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p} \otimes_A \Psi'_A(D) \arrow[r, "c_A(D)"] & \mathbf{d}_{A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}} D
\end{tikzcd}$$

where we used in the upper left corner the isomorphism

$$A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p} \otimes_{\varphi, A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}} A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p} \otimes_A \Psi'_A(D) \cong A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p} \otimes_A \Psi'_A(D),$$

and

(iii) an action of $\Gamma_{\mathbb{Q}_p}$, i.e. a homomorphism $g_A(D) : \Gamma_{\mathbb{Q}_p} \to \mathrm{Aut}_A(\Psi'_A(D))$, such that for all $\gamma \in \Gamma_{\mathbb{Q}_p}$ the following diagram commutes:

$$\begin{tikzcd}
A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p} \otimes_A \Psi'_A(D) \arrow[r, "c_A(D)"] \arrow[d, "1\otimes\gamma"] & A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p} \otimes_{\gamma, A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}} \mathbf{d}_{A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}} D \arrow[d, "\gamma"] \\
A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p} \otimes_A \Psi'_A(D) \arrow[r, "c_A(D)"] & \mathbf{d}_{A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}} D.
\end{tikzcd}$$

Remark 6.1.3. The existence and the uniqueness of primitive determinants of $(\varphi, \Gamma_{\mathbb{Q}_p})$-modules are not as clear as in Nakamura's case. We conjecture below the existence of good primitive determinants which interpolate Nakamura's Δ_2-objects (which are also defined below).

Construction 6.1.4. Assume for this construction that A is commutative. Denote with $\Psi^{\mathrm{Nak}}_A(D)$ a primitive determinant which lives in the graded line bundle determinant category, i.e. we utilise the same definition as before and just replace every occurrence of the determinant category $\mathbf{Det}^{(\mathrm{sh})}(A)$ by $\mathbf{Det}^{\mathrm{KM}}(A)$. In this situation Nakamura gives a general construction for $\Psi_A^{\mathrm{Nak}}(D)$ in [Nak13, p. 32] as follows: let D be a general $(\varphi, \Gamma_{\mathbb{Q}_p})$-module. Then $\det_{A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}} D$ is a rank one $(\varphi, \Gamma_{\mathbb{Q}_p})$-module, which according to theorem 5.2.11 is isomorphic to $(A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p})(\delta) \otimes_A \mathcal{L}$ where δ is a character $\mathbb{Q}_p^\times \to A^\times$. He defines

$$\Psi_A^{\mathrm{Nak}}(D) := \left(\operatorname{rk} D, \left\{ x \in \det_{A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}} D \;\middle|\; \begin{array}{l} \varphi(x) = \delta(p)x, \\ \gamma(x) = \delta(\chi(\gamma))x \text{ for } \gamma \in \Gamma_{\mathbb{Q}_p} \end{array} \right\} \right),$$

which is an object in $\mathbf{Det}^{\mathrm{KM}}(A)$. We note that $\operatorname{rk} D$ factors through $\operatorname{Spec} A$ as required due to the local Euler-Poincaré characteristic formula (see [KPX14, Thm. 4.4.5(2)]).

Furthermore we define φ and $\gamma \in \Gamma_{\mathbb{Q}_p}$ using δ, i.e. we set

$$\begin{aligned} \varphi &:= \delta(p) \in \operatorname{Aut}_A(\Psi_A^{\mathrm{Nak}}(D)) \qquad \text{and} \\ g_A(D)(\gamma) &:= \delta(\chi(\gamma)) \in \operatorname{Aut}_A(\Psi_A^{\mathrm{Nak}}(D)). \end{aligned}$$

Note that if the projection functor $\det_A^{\mathrm{KM}}$ from the universal to the graded line bundle determinant category over A is an equivalence of categories, then Nakamura's definition is our candidate for a good primitive determinant, i.e. $\Psi'_A(D) := \Psi_A^{\mathrm{Nak}}(D)$ together with the obvious choices for $c_A(D)$, φ and $g_A(D)$.

Remark 6.1.5. We are still in the setting of the last construction and we assume the equivalence of the determinant categories. Then, for a module whose rank is not 1 it is not clear if the operators φ and γ defined above fulfil 6.1.2(ii) and 6.1.2(iii) respectively. After applying the projection functor $\det^{\mathrm{KM}}_{A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}}$ the diagrams become commutative, however this projection functor might not be an equivalence of categories.

Nevertheless we expect the operators to fulfil the correct relations in general. Proving this however requires a better understanding of $K_1(A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p})$ which we currently lack.

We note that for triangulable $(\varphi, \Gamma_{\mathbb{Q}_p})$-modules the required statements for φ and γ can be shown.

Conjecture 6.1.6. *For A as in hypothesis 6.1.1(i) there exists a full subcategory $\boldsymbol{\varphi\Gamma}^{\mathrm{pdet}}_{\mathrm{rig},\mathbb{Q}_p,A}$ of the category $\boldsymbol{\varphi\Gamma}_{\mathrm{rig},\mathbb{Q}_p,A}$ of $(\varphi, \Gamma_{\mathbb{Q}_p})$-modules over $A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}$ which admit a primitive determinant $\Psi_A(D)$, where the $\Psi_A(D)$ satisfy the following compatibilities:*

(i) Let A' be as in hypothesis 6.1.1a) and let Y be an A'-A-bi-module which is a finitely generated, projective A'-module with a continuous commuting right A-action. Then

the $(\varphi, \Gamma_{\mathbb{Q}_p})$-module $Y \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p} \otimes_{A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}} D$ is an object of $\boldsymbol{\varphi\Gamma}^{\mathrm{pdet}}_{\mathrm{rig},\mathbb{Q}_p,A'}$ and

$$\Psi_{A'}(Y \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p} \otimes_{A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}} D) = Y \otimes_A \Psi_A(D)$$

which is compatible with $c_A(-)$, φ and $\gamma \in \Gamma_{\mathbb{Q}_p}$.

(ii) The construction behaves well with respect to short exact sequences, i.e. let

$$0 \longrightarrow D' \longrightarrow D \longrightarrow D'' \longrightarrow 0.$$

be a short exact sequence of objects in $\boldsymbol{\varphi\Gamma}^{\mathrm{pdet}}_{\mathrm{rig},\mathbb{Q}_p,A}$. Then

$$\Psi_A(D) = \Psi_A(D') \cdot \Psi_A(D'')$$

which is compatible with $c_A(-)$, φ and $\gamma \in \Gamma_{\mathbb{Q}_p}$.

(iii) $\Psi_A(D)$ commutes with Cartier duals, i.e.

$$\Psi_A(D^D) = \Psi_A(D)^*$$

which is compatible with $c_A(-)$, φ and $\gamma \in \Gamma_{\mathbb{Q}_p}$.

(iv) If A is a finite field extension over $\mathbb{Q}_p$, then construction 6.1.4 yields the primitive determinant $\Psi_A(D)$.[1]

(v) If D is a rank one module and A is commutative, then construction 6.1.4 yields the primitive determinant $\Psi_A(D)$.

(vi) Assume $A = A_n$ for $n \leq \infty$ (see section 3.5) and that there exists an A-submodule X_D of D such that the natural map

$$(A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}) \otimes_A X_D \longrightarrow D$$

is an isomorphism. Then

$$[\Psi_A(D)] = [X_D]$$

in $K_0^{(\mathrm{sh})}(A)$.

[1] See remark 6.1.5.

(vii) *Assume that Λ is as in proposition 3.6.21 and that T is a free rank one Λ-representation. Let $V = A \otimes_\Lambda T$, where $A = A_n$ ($n \leq \infty$) is an associated analytic space. Then there exists a canonical isomorphism*

$$b_A(T) : (A \mathbin{\widehat{\otimes}} \mathbb{Q}_p^{\mathrm{ur}}) \otimes_A \mathbf{d}_A V \xrightarrow{\sim} (A \mathbin{\widehat{\otimes}} \mathbb{Q}_p^{\mathrm{ur}}) \otimes_A \Psi_A(D)$$

such that the induced isomorphism

$$(A \mathbin{\widehat{\otimes}} \tilde{\mathbf{B}}^\dagger) \otimes_A \mathbf{d}_A V \xrightarrow{\sim} (A \mathbin{\widehat{\otimes}} \tilde{\mathbf{B}}^\dagger) \otimes_A \Psi_A(D) \xrightarrow{\sim} (A \mathbin{\widehat{\otimes}} \tilde{\mathbf{B}}^\dagger) \otimes_{A \mathbin{\widehat{\otimes}} \mathbf{B}_K^\dagger} \mathbf{d}_A \mathbf{D}^\dagger_{\mathrm{rig},\mathbb{Q}_p}(V)$$

comes from proposition 5.4.7(iii). Furthermore, in the commutative case $b_A(T)$ coincides with the isomorphism induced by the inclusion of $\Lambda_a \subset \mathbb{Z}_p^{\mathrm{ur}} \mathbin{\widehat{\otimes}} \Lambda$ (see [Nak13, Lem. 3.1]).

(viii) *Let D_1 and D_2 be objects in $\boldsymbol{\varphi\Gamma}^{\mathrm{pdet}}_{\mathrm{rig},\mathbb{Q}_p,A}$ such that*

$$\Psi_A(D_i) = \mathbf{d}_A V_i.$$

Then

$$\Psi_A(D_1 \otimes_{A \mathbin{\widehat{\otimes}} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}} D_2) = \mathbf{d}_A(V_1 \otimes_A V_2)$$

with the obvious induced $c_A(D_1 \otimes_{A \mathbin{\widehat{\otimes}} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}} D_2)$, φ and $\gamma \in \Gamma_{\mathbb{Q}_p}$.

Remark 6.1.7. Part (vi) is a consequence of the belief that the canonical map

$$K_0^{(\mathrm{sh})}(A) \to K_0^{(\mathrm{sh})}(A \mathbin{\widehat{\otimes}} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p})$$

is injective. This belief is supported by the work of Tamás Csige (see [Csi16]).

Part (vii) can be understood as a consequence Fukaya-Kato's local ϵ-isomorphism conjecture.

6.2 The Local ϵ-isomorphism Conjecture for $(\varphi, \Gamma_{\mathbb{Q}_p})$-modules

This section is an extension of the ideas of Nakamura as presented in [Nak13, §3.4]. Throughout the section we assume:

Hypothesis 6.2.1. Additionally to hypothesis 6.1.1 we assume that D is an object of

$\boldsymbol{\varphi\Gamma}^{\mathrm{pdet}}_{\mathrm{rig},\mathbb{Q}_p,A}$ with (locally)[2] perfect cohomology $\mathbf{R}\Gamma(\mathbb{Q}_p, D)$.

Definition 6.2.2. We define

$$\Delta_{A,1}(D) \coloneqq \mathbf{d}_A\, \mathbf{R}\Gamma(\mathbb{Q}_p, D)$$

which is possible as we assumed that $\mathbf{R}\Gamma(\mathbb{Q}_p, D)$ is perfect and

$$\Delta_{A,2}(D) \coloneqq \Psi_A(D).$$

Furthermore, we define the *fundamental line of D* by

$$\Delta_A(D) \coloneqq \Delta_{A,1}(D) \cdot \Delta_{A,2}(D).$$

Conjecture 6.2.3. *We assume conjecture 6.1.6. For every pair (A, D) which fulfils the requirements of hypothesis 6.2.1 there exists a compatible system of A-isomorphisms*

$$\epsilon_{A,\xi}(D) : \mathbf{1}_A \xrightarrow{\sim} \Delta_A(D),$$

where ξ is a basis of $\mathbb{Z}_p(1)$, satisfying the following properties:

(i) Let

$$0 \longrightarrow D' \longrightarrow D \longrightarrow D'' \longrightarrow 0$$

be a short exact sequence of $(\varphi, \Gamma_{\mathbb{Q}_p})$-modules which fulfil hypothesis 6.2.1, then the canonical isomorphisms

$$\begin{aligned} &\Delta_{A,1}(D) \xrightarrow{\sim} \Delta_{A,1}(D') \cdot \Delta_{A,1}(D'') && \textit{and} \\ &\Delta_{A,2}(D) \cong \Delta_{A,2}(D') \cdot \Delta_{A,2}(D'') && (6.1.6(ii)) \end{aligned}$$

yield a commutative diagram

$$\epsilon_{A,\xi}(D) = \epsilon_{A,\xi}(D') \cdot \epsilon_{A,\xi}(D'').$$

(ii) Let A' be as in hypothesis 6.2.1(i) and let Y be an A'-A-bi-module which is a finitely generated, projective A'-module with a continuous commuting right A-action, then

[2] See definition 2.6.3.

set $D' = Y \mathbin{\widehat{\otimes}} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p} \otimes_{A \mathbin{\widehat{\otimes}} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}} D$*. There are the two homomorphisms*

$$\begin{aligned} Y \otimes \Delta_{A,1}(D) &\longrightarrow \Delta_{A',1}(D') \qquad \textit{and} \\ Y \otimes \Delta_{A,2}(D) &\cong \Delta_{A',2}(D') \end{aligned}$$

where the second one is an isomorphism by conjecture 6.1.6 part (i) and we assume that the first homomorphism is an isomorphism. Then D' *also fulfils 6.2.1 and these two isomorphisms yield the commutative diagram*

$$Y \otimes_A \epsilon_{A,\xi}(D) = \epsilon_{A',\xi}(D').$$

(iii) *There are two homomorphisms*

$$\begin{aligned} \Psi(\mathbb{Q}_p, D) : \Delta_{A,1}(D^D)^*[-2] &\xrightarrow{\sim} \Delta_{A,1}(D) \qquad \textit{and} \\ \Delta_{A,2}(D^D) &\cong \Delta_{A,2}(D)^D, \end{aligned}$$

where the second one is an isomorphism by conjecture 6.1.6 part (iii) and we assume that the first homomorphism, which is defined in 5.9.3, is an isomorphism. (See [Ven07, §1] for the definition of the dual of a determinant.) Then

$$\epsilon_{A,\xi}(D) \cdot \epsilon_{A^{\mathrm{op}},\xi^{-1}}(D^D)^* \cdot \mathbf{d}_A\, \Psi(\mathbb{Q}_p, D) = \mathbf{d}_A(\xi : D(1) \to D).$$

(iv) *For* $\gamma \in \Gamma_{\mathbb{Q}_p}$ *we have*

$$\epsilon_{A,\gamma\xi}(D) = g_A(D)(\gamma) \cdot \epsilon_{A,\xi}(D).$$

(v) *Let* $A = F$ *be a finite extension of* $\mathbb{Q}_p$ *and assume that* D *is deRham. Then* $\epsilon_{F,\xi}(D)$ *is* $\epsilon^{\mathrm{dR}}_{F,\xi}(D)$*, which is defined in [Nak13, §3.3].*

(vi) *In the notation of conjecture 6.1.6(vii), assuming* (A, V) *comes from* (Λ, T)*, the* ϵ*-isomorphism given by conjecture [FK06, Conj. 3.4.3] is compatible with* $\epsilon_{A_n,\xi}(D)$*, i.e.*

$$A_n \mathbin{\widehat{\otimes}} \mathbb{Q}_p^{\mathrm{ur}} \otimes_{\Lambda \mathbin{\widehat{\otimes}} \mathbb{Z}_p^{\mathrm{ur}}} \epsilon_{\Lambda,\xi}(T) = \left(A_n \mathbin{\widehat{\otimes}} \mathbb{Q}_p^{\mathrm{ur}} \otimes_{A_n} \epsilon_{A_n,\xi}(D)\right) \circ b_{A_n}(T).$$

7 p-adic L-Functions

We closely follow [FK06, §4.1].

We assume the following:

Hypothesis 7.0.1. (i) In the notation of §3.5 let $\Lambda = \mathcal{O}_K[[G]]$ be the Iwasawa algebra of a compact p-adic Lie group G and let A be an $A_n(G)$ for $2 \le n \le \infty$.

(ii) Let T be a finitely generated projective Λ-module with a continuous $\Lambda[\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})]$-module structure[1] such that the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action on T is unramified at almost all primes and set $V = A \otimes_\Lambda T$.

(iii) Let D^0 be a $(\varphi, \Gamma_{\mathbb{Q}_p})$-module together with a map

$$\mathrm{lc}_{D^0} : D^0 \longrightarrow \mathbf{D}^\dagger_{\mathrm{rig},\mathbb{Q}_p}(V)$$

subject to the following restrictions:

a) D^0 is an object of $\boldsymbol{\varphi\Gamma}^{\mathrm{pdet}}_{\mathrm{rig},\mathbb{Q}_p,A}$,

b) $\mathbf{R}\Gamma(\mathbb{Q}_p, D^0)$ is (locally)[2] perfect, and

c) there exists an isomorphism

$$\mathbf{d}_A V^+ \xrightarrow{\sim} \Delta_{A,2}(D^0),$$

here $(-)^+$ are the invariants under the complex conjugation.

Remark 7.0.2. In the case $n < \infty$ we use the determinant category $\mathbf{Det}(A)$, the notion of perfect complexes and the normal K-groups $K_i(A)$. If $n = \infty$ we use $\mathbf{Det}^{\mathrm{sh}}(A)$, locally perfect complexes and $K_i^{\mathrm{sh}}(A)$.

Remark 7.0.3. Usually one requires D^0 to be a direct summand of $\mathbf{D}^\dagger_{\mathrm{rig},\mathbb{Q}_p}(V)$ but there are applications where we can usefully exploit the more general setting described above, see chapter 9.

[1] see definition 3.6.3

[2] See definition 2.6.3.

7.1 p-adic L-Functions for p-adic Galois Representations

Definition 7.1.1. Fix a non-empty open subset U of $\operatorname{Spec}\mathbb{Z}[1/p]$ on which the action of $\mathrm{Gal}_{\mathbb{Q}}$ on T is unramified. We define the *Selmer complex* $SC(U, V, \mathrm{lc}_{D^0})$ as the mapping fibre of (see [Pot13, §1.5])

$$C^\bullet_{\mathrm{cts}}(U, V) \oplus C^\bullet_{\varphi,\gamma_{\mathbb{Q}_p}}(D^0) \longrightarrow \bigoplus_{\ell \notin U} C^\bullet_{\mathrm{cts}}(\mathbb{Q}_\ell, V)$$

where the map of $C^\bullet_{\varphi,\gamma_{\mathbb{Q}_p}}(D^0)$ to $C_{\mathrm{cts}}(\mathbb{Q}_p, V)$ is given via

$$C^\bullet_{\varphi,\gamma_{\mathbb{Q}_p}}(D^0) \xrightarrow{C^\bullet_{\varphi,\gamma_{\mathbb{Q}_p}}(\mathrm{lc}_{D^0})} C^\bullet_{\varphi,\gamma_{\mathbb{Q}_p}}(\mathbb{Q}_p, \mathbf{D}^\dagger_{\mathrm{rig}}(V)) \xrightarrow{\sim} C^\bullet_{\mathrm{cts}}(\mathbb{Q}_p, V),$$

where the quasi-isomorphism used here is the result of theorem 5.8.3 or remark 5.10.9 respectively.

Lemma 7.1.2. *We have a distinguished triangle*

$$\mathbf{R}\Gamma_{\mathrm{cts},c}(U, V) \longrightarrow SC(U, V, \mathrm{lc}_{D^0}) \longrightarrow V^+ \oplus \mathbf{R}\Gamma(\mathbb{Q}_p, D^0) \longrightarrow .$$

Proof. We have the following diagram of distinguished triangles

$$\begin{tikzcd}
\mathbf{R}\Gamma_{\mathrm{cts},c}(U,V) \arrow[r] \arrow[d] & SC(U,V,\mathrm{lc}_{D^0}) \arrow[r] \arrow[d] & V^+ \oplus \mathbf{R}\Gamma(\mathbb{Q}_p, D^0) \arrow[r] \arrow[d] & {} \\
\mathbf{R}\Gamma_{\mathrm{cts}}(U,V) \arrow[r] \arrow[d] & \mathbf{R}\Gamma_{\mathrm{cts}}(U,V) \oplus \mathbf{R}\Gamma(\mathbb{Q}_p, D^0) \arrow[r] \arrow[d] & \mathbf{R}\Gamma(\mathbb{Q}_p, D^0) \arrow[r] \arrow[d, "0"] & {} \\
\bigoplus_{\nu \notin U} \mathbf{R}\Gamma_{\mathrm{cts}}(\mathbb{Q}_\nu, V) \arrow[r] \arrow[d] & \bigoplus_{\ell \notin U} \mathbf{R}\Gamma_{\mathrm{cts}}(\mathbb{Q}_\ell, V) \arrow[r, "0"] \arrow[d] & \mathbf{R}\Gamma(\mathbb{R}, V)[1] \arrow[r] \arrow[d] & {} \\
{} & {} & {} &
\end{tikzcd}$$

by [Wei94, Ex. 10.2.6]. □

Corollary 7.1.3. *Assuming conjecture 6.2.3 we deduce* $[SC(U, V, \mathrm{lc}_{D^0})] = 0$ *in* $K_0^{(\mathrm{sh})}(A)$.

Proof. The claim follows from [FK06, Prop. 2.1.3], which states that $\mathbf{R}\Gamma_{\mathrm{cts},c}(U, V)$ has the trivial class in $K_0^{(\mathrm{sh})}(A)$, and hypothesis 7.0.1(iii)c) together with the ϵ-isomorphism of conjecture 6.2.3 which state that $V^+ \oplus \mathbf{R}\Gamma(\mathbb{Q}_p, D^0)$, also has the trivial class in $K_0^{(\mathrm{sh})}(A)$. □

Definition 7.1.4. Fix an isomorphism

$$\beta : \tilde{A} \otimes_A \mathbf{d}_A V^+ \xrightarrow{\sim} \tilde{A} \otimes_A \Delta_{A,2}(D^0)$$

where $\tilde{A} := \mathbb{Q}_p^{\mathrm{ur}} \widehat{\otimes} A$ if we are in the case $n < \infty$ and $\tilde{A} := \tilde{A}_\infty := \varprojlim_n \mathbb{Q}_p^{\mathrm{ur}} \widehat{\otimes} A_n$ if we are in the case $n = \infty$. We note that $\tilde{A}_\infty$ is also a Fréchet-Stein algebra (see [ST03, Thm. 5.1]), hence §2.6 still applies and we can form groups like $K_1^{\mathrm{sh}}(\tilde{A}_\infty)$. Let $\Sigma(U, V, \mathrm{lc}_{D^0})$ be the smallest full subcategory of $\mathcal{P}(A)$, the category of (locally) perfect complexes, satisfying the conditions (i)-(iv) in [FK06, §1.3.1] and containing all objects that are quasi-isomorphic to $SC(U, V, \mathrm{lc}_{D^0})$.

Consider the isomorphism

$$\zeta_\Lambda(T) : \mathbf{1}_\Lambda \xrightarrow{\sim} \mathbf{d}_\Lambda \, \mathbf{R}\Gamma_{\mathrm{cts},c}(U, T)^{-1}$$

from the ζ-isomorphism conjecture [FK06, Conj. 2.3.2] which induces the isomorphism

$$\zeta_A(V) : \mathbf{1}_A \xrightarrow{\sim} \mathbf{d}_A \, \mathbf{R}\Gamma_{\mathrm{cts},c}(U, V)^{-1}$$

by the base change theorem (see corollary 3.6.24). Furthermore, by the ϵ-isomorphism conjecture 6.2.3 there is the following isomorphism

$$\epsilon_{A,\xi}(D^0) : \mathbf{1}_A \xrightarrow{\sim} \Delta_A(D^0).$$

The above distinguished triangle induces the isomorphism

$$\mathbf{d}_A \, \mathbf{R}\Gamma_{\mathrm{cts},c}(U, V)^{-1} \cong \mathbf{d}_A \, SC(U, V, \mathrm{lc}_{D^0})^{-1} \cdot \mathbf{d}_A \, V^+ \cdot \mathbf{d}_A \, \mathbf{R}\Gamma(\mathbb{Q}_p, D^0).$$

Then it follows that the product

$$\zeta_\beta(U, T, \mathrm{lc}_{D^0}) := \beta \cdot (\epsilon_{A,\xi}^{-1}(D^0))_{\tilde{A}} \cdot \zeta_A(V)_{\tilde{A}}$$

is an element of

$$\mathrm{Isom}(\mathbf{1}_A \to \mathbf{d}_A \, SC(U, V, \mathrm{lc}_{D^0})^{-1}) \times^{K_1^{(\mathrm{sh})}(A)} K_1^{(\mathrm{sh})}(\tilde{A}).$$

We denote the class in $K_1^{(\mathrm{sh})}(A, \Sigma(U, V, \mathrm{lc}_{D^0})) \times^{K_1^{(\mathrm{sh})}(A)} K_1^{(\mathrm{sh})}(\tilde{A})$ by the same symbol and call it the *p-adic ζ-function of the pair (T, lc_{D^0}) with respect to U and β*. (We suppress the dependency on ξ.)

When lc_{D^0} is injective we also write $SC(U, V, D^0)$ and $\zeta_\beta(U, T, D^0)$.

Remark 7.1.5. We expect that the ζ-isomorphism $\zeta_A(V)$ has its own right to exist, regardless of whether V comes from a T as in hypothesis 7.0.1. The situation should mirror the situation with ϵ-isomorphisms, where conjecture 6.2.3 generalises Fukaya-Kato's ϵ-conjecture [FK06, Conj. 3.4.3] to a certain class of nc-affinoids.

Remark 7.1.6. We have $\zeta_{a\beta}(U, T, \mathrm{lc}_{D^0}) = a\zeta_\beta(U, T, \mathrm{lc}_{D^0})$ for $a \in K_1^{(\mathrm{sh})}(\tilde{A})$.

Remark 7.1.7. Let $(\Lambda, A, T, \mathrm{lc}_{D^0})$ and

$$(\Lambda', A', T', \mathrm{lc}_{D^{0,\prime}}) = (\Lambda', A', Y \otimes_\Lambda T, Y_{A'} \widehat{\otimes}_A \mathrm{lc}_{D^0})$$

be a quadruple which fulfil hypothesis 7.0.1 where Y is Λ'-Λ-bi-module such that Y is a finitely generated, projective, right Λ'-module and the left action of Λ on Y is continuous and commutes with the right action of Λ'. Furthermore we assume that the same holds for $Y_{A'} := A' \otimes_{\Lambda'} Y$ with Λ replaced by A and Λ' replaced by A'. Then, assuming that the canonical map

$$Y_{A'} \otimes_A^L \mathbf{R}\Gamma(K, D^0) \longrightarrow \mathbf{R}\Gamma(K, D^{0,\prime})$$

is an isomorphism, we have the canonical isomorphism

$$Y_{A'} \otimes_A SC(U, V, \mathrm{lc}_{D^0}) \longrightarrow SC(U, V', \mathrm{lc}_{D^{0,\prime}})$$

and $Y_{A'} \otimes_A -$ sends $\zeta_\beta(U, T, \mathrm{lc}_{D^0})$ to $\zeta_{\beta'}(U, T', \mathrm{lc}_{D^{0,\prime}})$ assuming that β' is induced by the original β.

7.2 Values of p-adic L-Functions

Now we define the evaluation of the p-adic L-function at certain homomorphisms:

Definition 7.2.1. Let $L/\mathbb{Q}_p$ a finite extension, let $n \geq 1$ and let $\rho : A \to M_n(L)$ be a continuous ring homomorphism which factors through A_m for $m \gg 0$ such that $L^n \otimes_A^L SC(U, V, \mathrm{lc}_{D^0})$ is *acyclic*, where L^n is regarded as row vectors and A acts on it from the right via ρ.

For any object $C \in \Sigma(U, V, \mathrm{lc}_{D^0})$, $L^n \otimes_A^L C$ becomes acyclic: the full subcategory of $\mathcal{P}(A)$ which contains all objects which become acyclic after applying $L^n \otimes_A^L -$, contains $SC(U, V, \mathrm{lc}_{D^0})$ and fulfils (i)-(iv) in [FK06, §1.3.1]. Hence, this category contains $\Sigma(U, V, \mathrm{lc}_{D^0})$ since this is the smallest category with these properties.

Thus, $L^n \otimes_A^L -$ induces a homomorphism

$$K_1^{(\mathrm{sh})}(A, \Sigma(U, V, \mathrm{lc}_{D^0})) \times^{K_1^{(\mathrm{sh})}(A)} K_1^{(\mathrm{sh})}(\tilde{A}) \to K_1(\tilde{L}) \to K_1(\hat{L}^{\mathrm{ur}}) = (\hat{L}^{\mathrm{ur}})^\times.$$

Here, $\hat{L}^{\mathrm{ur}}$ is the completion of the maximal unramified extension of L. We call the image of $\zeta_\beta(U, T, \mathrm{lc}_{D^0})$ in $(\hat{L}^{\mathrm{ur}})^\times$ the value at ρ and denote it by $\zeta_\beta(U, T, \mathrm{lc}_{D^0})(\rho)$.

We will now try to understand when the acyclicity condition from above is satisfied.

Lemma 7.2.2 ([FK06, Lem. 4.1.6])**.** *Let L be a finite extension of $\mathbb{Q}_p$. Let ℓ be a prime not equal to p, V be finite-dimensional vector space endowed with a continuous action of* $\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ *and let the Frobenius polynomial of V be*

$$P_{L,\ell}(V, u) := \det_L(1 - \varphi_\ell u | V^{I_\ell}) \in L[u]$$

where φ_ℓ is the geometric *Frobenius. Then the following conditions are equivalent:*

(i) The polynomial $P_{L,\ell}(V, u)$ does not have a zero at $u = 1$.

(ii) $H^0(\mathbb{Q}_\ell, V)$ vanishes.

(iii) The complex $C_f^\bullet(\mathbb{Q}_\ell, V)$, defined in [FK06, p. 33], is acyclic.

Proof. For the equivalence of (i) and (ii) note that $H^0(\mathbb{Q}_\ell, V) = (V^{I_\ell})^{\varphi_\ell = 1}$. The equivalence of (i) and (iii) is also clear after inspection of the definition of $C_f^\bullet(\mathbb{Q}_\ell, V)$. □

The following lemma is a generalisation of [FK06, Lem. 4.1.7], in a similar guise it can also be found in [Pot13, Prop. 3.7].

Lemma 7.2.3. *Let D be a deRham $(\varphi, \Gamma_{\mathbb{Q}_p})$-module over $L \otimes \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}$ (i.e. $A = L$) with L a finite extension over $\mathbb{Q}_p$ and let the Frobenius polynomial of D be*

$$P_{L,p}(D, u) := \det_L(1 - \varphi u | \mathbf{D}_{\mathrm{crys}}(D))$$

(regarding $\mathbf{D}_{\mathrm{crys}}(D)$ see definition 5.6.1). Let F be a $(\varphi, \Gamma_{\mathbb{Q}_p})$-direct summand of D such that

$$\mathbf{D}_{\mathrm{dR}}(F) \cong \mathbf{D}_{\mathrm{dR}}(D)/\mathbf{D}^0_{\mathrm{dR}}(D).$$

Then (i) and (ii) are equivalent:

(i) The polynomials $P_{L,p}(D, u)P_{L,p}(F, u)^{-1}$ and $P_{L,p}(F^D, u)$ do not have a zero at $u = 1$.

(ii) $H^0(\mathbb{Q}_p, D/F) = 0$ *and* $H^0(\mathbb{Q}_p, F^D) = 0$.

Both imply

(iii) $\mathbf{R}\Gamma(\mathbb{Q}_p, F) \xleftarrow{\sim} \mathbf{R}\Gamma_f(\mathbb{Q}_p, F) \xrightarrow{\sim} \mathbf{R}\Gamma_f(\mathbb{Q}_p, D)$ *(regarding* $\mathbf{R}\Gamma_f(\mathbb{Q}_p, -)$ *see definition 5.6.3).*

Proof. To show the equivalence of (i) and (ii) note that

$$P_{L,p}(D, u)P_{L,p}(F, u)^{-1} = P_{L,p}(D/F, u)$$

as F is a direct summand of D and hence the corresponding $\mathbf{B}_{\mathrm{crys}}$-sequence is split exact. Furthermore, note that for $(\varphi, \Gamma_{\mathbb{Q}_p})$-modules X we have $H^0(\mathbb{Q}_p, X) = (X^\Gamma)^{\varphi=1} = \mathbf{D}^+_{\mathrm{crys}}(X)^{\varphi=1}$. Additionally, we have $\mathbf{D}^+_{\mathrm{crys}}(X) = \mathbf{D}_{\mathrm{crys}}(X)$ if X fulfils $\mathbf{D}_{\mathrm{dR}}(X) = \mathbf{D}^0_{\mathrm{dR}}(X)$, i.e. has only non-negative Hodge-Tate weights, which is true for D/F by assumption and F^D by duality (see [Nak13, p. 23, eq. (13)]).

We show the first arrow of (iii) and we essentially follow [Pot13, Prop. 3.7]: we have equality of cohomology outside of degree 1 and 2. Let us look at degree 1:

$$\begin{aligned}
\dim_L H^1(\mathbb{Q}_p, F) &\overset{(1)}{=} \dim_L H^0(\mathbb{Q}_p, F) + \dim_L H^2(\mathbb{Q}_p, F) + \operatorname{rk} F \\
&\overset{(2)}{=} \dim_L H^0(\mathbb{Q}_p, F) + \operatorname{rk} F \\
&\overset{(3)}{=} \dim_L H^0(\mathbb{Q}_p, F) + \dim_L \mathbf{D}_{\mathrm{dR}}(F) - \dim_L \mathbf{D}^0_{\mathrm{dR}}(F) \\
&\overset{(4)}{=} \dim_L H^1_f(\mathbb{Q}_p, F)
\end{aligned}$$

where we used:

(1) the local Euler-Poincaré characteristic formula [KPX14, Thm. 2.3.11(2)],

(2) Tate local duality $H^2(\mathbb{Q}_p, F) \cong H^0(\mathbb{Q}_p, F^D)^* = 0$, see [KPX14, Thm. 2.3.11(3)],

(3) F is deRham as D is deRham (and a direct summand) as well as $\mathbf{D}^0_{\mathrm{dR}}(F) = 0$, and

(4) remark 5.6.5.

Calculation (2) also shows that the cohomology coincides in degree 2. Hence, the first arrow is a quasi-isomorphism.

Regarding the second arrow, we get equality of the cohomology outside of degrees 0 and

1. We get the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{D}_{\mathrm{crys}}(F) \arrow[r] \arrow[d, "f_F"] & \mathbf{D}_{\mathrm{crys}}(D) \arrow[r] \arrow[d, "f_D"] & \mathbf{D}_{\mathrm{crys}}(D/F) \arrow[r] \arrow[d, "f_{D/F}"] & 0 \\
0 \arrow[r] & \mathbf{D}_{\mathrm{crys}}(F) \oplus t_{\mathrm{dR}}(F) \arrow[r] & \mathbf{D}_{\mathrm{crys}}(D) \oplus t_{\mathrm{dR}}(D) \arrow[r] & \mathbf{D}_{\mathrm{crys}}(D/F) \oplus t_{\mathrm{dR}}(D/F) \arrow[r] & 0
\end{tikzcd}$$

where the kernel of $f_{D/F}$ is $H^0(\mathbb{Q}_p, D/F)$ which vanishes. Hence, degree 0 is settled. It also follows that $H^1(\mathbb{Q}_p, F) \to H^1(\mathbb{Q}_p, D)$ is injective, so it suffices to compare the dimensions of $H^1_f(\mathbb{Q}_p, F)$ and $H^1_f(\mathbb{Q}_p, D)$: note that we assumed

$$t_{\mathrm{dR}}(F) = \mathbf{D}_{\mathrm{dR}}(F) \cong t_{\mathrm{dR}}(D).$$

□

The next lemma is a straightforward generalisation of [FK06, Lem. 4.1.8]:

Lemma 7.2.4. *Let L be a finite extension of $\mathbb{Q}_p$, V be a finite-dimensional L-vector space endowed with a continuous action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which is unramified at almost all primes, and let F be a $(\varphi, \Gamma_{\mathbb{Q}_p})$-module over $L \otimes \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}$ which is a direct summand of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$. Assume that the following conditions (i)–(v) are satisfied:*

(i) *V is a deRham representation of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$.*

(ii) $\mathbf{D}_{\mathrm{dR}}(F) \cong \mathbf{D}_{\mathrm{dR}}(V)/\mathbf{D}^0_{\mathrm{dR}}(V)$.

(iii) $H^0(\mathbb{Q}, V)$, $H^1_f(\mathbb{Q}, V)$, $H^0(\mathbb{Q}, V^D)$, $H^1_f(\mathbb{Q}, V^D)$ *all vanish.*

(iv) *For any prime $\ell \neq p$ that is not contained in U, $P_{L,\ell}(V, u)$ does not have a zero at $u = 1$.*

(v) *The polynomials $P_{L,p}(V, u)P_{L,p}(F, u)^{-1}$ and $P_{L,p}(F^D, u)$ do not have a zero at $u = 1$.*

Then:

$$H^m(U, V) \oplus H^m(\mathbb{Q}_p, F) \xrightarrow{\sim} \bigoplus_{\ell \notin U} H^m(\mathbb{Q}_\ell, V).$$

Proof. By lemma 7.2.2 and condition (iv) it follows that $C_f(\mathbb{Q}_\ell, V)$ is acyclic for $\ell \notin U \cup \{p\}$. Lemma 7.2.3 and conditions (i), (ii) and (v) show that $\mathbf{R}\Gamma(\mathbb{Q}_p, F) \xrightarrow{\sim} \mathbf{R}\Gamma_f(\mathbb{Q}_p, V)$.

Hence, the mapping fibre of the morphism

$$C^\bullet_{\mathrm{cts}}(U,V) \oplus C^\bullet_{\varphi,\gamma_{\mathbb{Q}_p}}(\mathbb{Q}_p,F) \to \bigoplus_{\ell \notin U} C^\bullet_{\mathrm{cts}}(\mathbb{Q}_\ell, V)$$

is quasi-isomorphic to the mapping fibre of

$$C^\bullet_{\mathrm{cts}}(U,V) \oplus C^\bullet_f(\mathbb{Q}_p,V) \to C^\bullet_{\mathrm{cts}}(\mathbb{Q}_p,V) \oplus \bigoplus_{\ell \notin U\cup\{p\}} C^\bullet_{\mathrm{cts}}(\mathbb{Q}_\ell, V)/C_f(\mathbb{Q}_\ell,V)$$

as $C_f(\mathbb{Q}_\ell,V)$ is quasi-isomorphic to 0, which in turn identifies with the mapping fibre of

$$C^\bullet_{\mathrm{cts}}(U,V) \to \bigoplus_{\ell \notin U} C^\bullet_{\mathrm{cts}}(\mathbb{Q}_\ell, V)/C_f(\mathbb{Q}_\ell,V),$$

in the derived category, which is just $\mathbf{R}\Gamma_f(\mathbb{Q},V)$ (see [FK06, p. 33]). By (iii) it now follows that $\mathbf{R}\Gamma_f(\mathbb{Q},V)$ is acyclic (see [FK06, §2.4.3, eq. 2.6]). □

Corollary 7.2.5. *Let L be a finite extension of $\mathbb{Q}_p$, $n \geq 1$ and let $\rho : A \to M_n(L)$ be a continuous ring homomorphism which has an integral model which fulfils hypothesis 3.6.17 and in the case of $A = A_\infty$ factors through some A_m for $m < \infty$. Assume hypothesis 7.0.1. Define a finite-dimensional L-vector space*

$$V_\rho \coloneqq L^n \otimes_A V$$

endowed with a continuous action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and a $(\varphi,\Gamma_{\mathbb{Q}_p})$-module

$$D^0_\rho \coloneqq (L \otimes \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p})^n \otimes_{A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}} D^0$$

over $L \otimes \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}$. Assume that D^0_ρ is a direct summand of $\mathbf{D}^\dagger_{\mathrm{rig}}(V_\rho)$ via lc_{D^0}. As before, elements of L^n are regarded as row vectors and A acts on from the right via ρ. Assume that the pair (V_ρ, D^0_ρ) satisfies condition (i)–(v) of the previous lemma. Then $L^n \otimes_A SC(U,V,\mathrm{lc}_{D^0})$ is acyclic.

Proof. Note that

$$\begin{aligned} L^n \otimes^L_A \mathbf{R}\Gamma(U,V) &\overset{(1)}{\cong} \mathbf{R}\Gamma(U,V_\rho), \\ L^n \otimes^L_A \mathbf{R}\Gamma(\mathbb{Q}_\ell,V) &\overset{(2)}{\cong} \mathbf{R}\Gamma(\mathbb{Q}_\ell,V_\rho), \\ L^n \otimes^L_A \mathbf{R}\Gamma(\mathbb{Q}_p,D^0) &\overset{(3)}{\cong} \mathbf{R}\Gamma(\mathbb{Q}_p,D^0_\rho) \end{aligned}$$

where (1) and (2) are due to theorem 3.6.20 and (3) is due to proposition 5.7.6. Hence, the statement follows from lemma 7.2.4. □

Remark 7.2.6. Assume that D^0_ρ is not a saturated subspace of $\mathbf{D}^\dagger_{\mathrm{rig}}(V_\rho)$, e.g. $t^{-k}D^0_\rho$ is a direct summand of $\mathbf{D}^\dagger_{\mathrm{rig}}(V_\rho)$, but $L^n \otimes_A SC(U, V, \mathrm{lc}_{D^0})$ might not be acyclic. However using the ideas of the proof of proposition 5.7.8 one can create a canonical trivialisation of the above complex. Hence, also in this case we can associate a value to the p-adic L-function. We leave the details to the reader.

7.3 Review of the Theory of Mixed Motives with Coefficients

Since we cannot cover the too vast a topic of the philosophy and the desired properties of motives we contend ourselves with a quick review of the theory of mixed motives following [FPR94] and [Del79].

By a *motive over a finite extension F over $\mathbb{Q}$* we mean its collection of realisations as follows:

(i) The *deRham realisation M_{dR}* which is a finite dimensional F-vector space endowed with a descending Hodge filtration,

(ii) the *Betti realisation $M_{B,\mathfrak{p}}$* for every $\mathfrak{p} \in S_\infty(F)$ which is a finite dimensional $\mathbb{Q}$-vector space together with an action of the absolute Galois group of $F_\mathfrak{p}$, and

(iii) the *ℓ-adic realisation M_ℓ* which is a free $\mathbb{Q}_\ell$-vector space endowed with a continuous action of the absolute Galois group of F.

Furthermore we require the existence of comparison isomorphisms and of the weight filtrations as described in [FPR94, §2.1.1].

A *motive over a finite extension F over $\mathbb{Q}$ with an action of a finite extension of K over $\mathbb{Q}$* is a motive M over F as above together with a homomorphism

$$K \to \mathrm{End}(M).$$

7.4 Values of p-adic L-Functions at Motivic Points

Hypothesis 7.4.1. Let L and ρ be as in corollary 7.2.5. Assume that we have K-motive M over $\mathbb{Q}$ with K a finite extension of $\mathbb{Q}$, a homomorphism $K \to L$ and an isomorphism

$V_\rho \cong L \otimes_{\mathbb{Q}_p \otimes_{\mathbb{Q}} K} M_p$ as representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ over L. Fix an embedding of K into $\mathbb{C}$. Also assume that lc_{D^0} induces an isomorphism

$$\mathrm{lc}_{D^0}(\rho) : \mathbf{D}_{\mathrm{dR}}(D^0_\rho) \xrightarrow{\sim} \mathbf{D}_{\mathrm{dR}}(V_\rho)/\mathbf{D}^0_{\mathrm{dR}}(V_\rho)$$

Definition 7.4.2. Let L, ρ, K and M be as above, and fix embeddings $K \to \mathbb{C}$ and $L \to \bar{\mathbb{Q}}_p$. We assume that $H^0(\mathbb{Q}, M)$, $H^1_f(\mathbb{Q}, M)$, $H^0(\mathbb{Q}, M^D)$, $H^1_f(\mathbb{Q}, M^D)$ all vanish which implies condition (iii) of lemma 7.2.4.

Fix a K-basis γ of M_B^+ and a K-basis δ of t_M. We define

$$\begin{aligned} \Omega_{\infty,\gamma,\delta}(M) &\in \mathbb{C}^\times \qquad \text{and} \\ \Omega_{p,\beta,\gamma,\delta}(M, \mathrm{lc}_{D^0}) &\in (\hat{L}^{\mathrm{ur}})^\times. \end{aligned}$$

First, let $\Omega_{\infty,\gamma,\delta}(M)$ be the determinant of the period map $\mathbb{C} \otimes_K M_B^+ \xrightarrow{\sim} \mathbb{C} \otimes_K t_M$ with respect to γ and δ. This value has conjecturally a relation to L-values.

Next, $\Omega_{p,\beta,\gamma,\delta}(M, \mathrm{lc}_{D^0})$. We have isomorphisms

$$\theta^{-1}_{dR,L,\xi}(D^0_\rho) : \Delta_{L,2}(D^0_\rho) \xrightarrow{\sim} \mathbf{d}_L \, \mathbf{D}_{\mathrm{dR}}(D^0_\rho), \tag{7.4.1}$$

$$\mathbf{d}_L \, \mathbf{D}_{\mathrm{dR}}(D^0_\rho) \xrightarrow{\sim} L \otimes_K \mathbf{d}_K \, t_M, \tag{7.4.2}$$

$$\mathbf{d}_K \, t_M \xrightarrow{\sim} \mathbf{d}_K \, M_B^+ \qquad \text{and} \tag{7.4.3}$$

$$\beta_\rho : \tilde{L} \otimes_K \mathbf{d}_K \, M_B^+ \xrightarrow{\sim} \tilde{L} \otimes_L \Delta_{L,2}(D^0_\rho). \tag{7.4.4}$$

The first isomorphism is defined in [Nak13, p. 38]. The second isomorphism comes from $\mathbf{D}_{\mathrm{dR}}(D^0_\rho) \cong \mathbf{D}_{\mathrm{dR}}(V_\rho)/\mathbf{D}^0_{\mathrm{dR}}(V_\rho)$ which depends on lc_{D^0}. The third isomorphism is induced by γ and δ. The fourth isomorphism is induced by β.

Multiplying these four isomorphisms gives an automorphism, i.e. an an element of $K_1(\tilde{L})$, which we can map to an element of $K_1(\hat{L}^{\mathrm{ur}})$.

Note that changing γ and δ changes both periods, however the change is in a complementary way so that their "ratio" is constant.

Lemma 7.4.3. *Let $a \in A$, then*

$$\Omega_{p,\beta,\gamma,\delta}(M, a \cdot \mathrm{lc}_{D^0}) = \det \rho(a) \cdot \Omega_{p,\beta,\gamma,\delta}(M, \mathrm{lc}_{D^0})$$

if $\det \rho(a) \neq 0$.

Proof. When going from lc_{D^0} to $a \cdot \mathrm{lc}_{D^0}$ equation (7.4.2) picks up a $\rho(a)$. □

Our analogue of Fukaya-Kato's main conjecture [FK06, Thm. 4.1.12] is the following

Theorem 7.4.4. *We assume that Λ, T and lc_{D^0} fulfil 7.0.1 and that in our situation Fukaya-Kato's ζ-isomorphism conjecture [FK06, Conj. 2.3.2] and the ϵ-isomorphism conjecture 6.2.3 are valid.*

Then the following holds:

(i) The boundary map

$$K_1^{(\mathrm{sh})}(A, \Sigma(U, V, \mathrm{lc}_{D^0})) \times^{K_1(A)} K_1(\tilde{A}) \to K_0^{(\mathrm{sh})}(\Sigma(U, V, \mathrm{lc}_{D^0}))$$

maps $\zeta_\beta(U, T, \mathrm{lc}_{D^0})$ to $[[SC(U, V, \mathrm{lc}_{D^0})]]$.

(ii) Let L, ρ, K and M be as in definition 7.4.2 and assume that (V_ρ, D^0_ρ) satisfies the conditions (ii)–(v) in lemma 7.2.4. Let $h(r) = \dim_K \mathrm{gr}^r M_{dR}$. Then $L_K(M, s)$ has neither a zero nor a pole at $s = 0$ and the value of $\zeta_\beta(U, T, \mathrm{lc}_{D^0})$ at ρ can be calculated to be

$$\left\{\frac{L_K(M,0)}{\Omega_\infty(M)}\right\} \cdot \Omega_p(M) \cdot \prod_{r\geq 1} \Gamma(r)^{h(-r)} \cdot \left\{\frac{P_{L,p}(V_\rho, u)}{P_{L,p}(D^0_\rho, u)}\right\}_{u=1} \cdot P_{L,p}((D^0_\rho)^D, 1) \cdot \prod_{\ell \notin U \cup \{p\}} P_{L,\ell}(V_\rho, 1).$$

Proof. Part (i) is clear, so we only have treat part (ii).

We have:

$$\tilde{L} \otimes_L \mathbf{1}_L \xrightarrow{\zeta_\beta(U,T,\mathrm{lc}_{D^0})_\rho} \tilde{L} \otimes_L \mathbf{d}_L \left(L^n \otimes_A SC(U, V, \mathrm{lc}_{D^0})\right)^{-1}.$$

Note that

$$\zeta_\beta(U, T, \mathrm{lc}_{D^0})(\rho) = \beta_\rho \cdot (\epsilon^{-1}_{A,\xi}(F))_\rho \cdot \zeta_A(V)_\rho = \beta_\rho \cdot \epsilon_{L,\xi}(D^0_\rho)^{-1}_{\tilde{L}} \cdot \zeta_L(V_\rho)_{\tilde{L}}$$

because of the base change properties of ϵ and ζ. Furthermore, note that the right hand side of the isomorphism is acyclic by corollary 7.2.5 via this construction:

$$\begin{aligned}
\mathbf{d}_L\left(L^n \otimes^L_A SC(U, V, \mathrm{lc}_{D^0})\right) &\overset{\text{q.i.}}{=} \mathbf{d}_L\left(L^n \otimes^L_A \mathbf{R}\Gamma_{\mathrm{cts}}(U, V)\right) \cdot \mathbf{d}_L\left(L^n \otimes^L_A \mathbf{R}\Gamma(\mathbb{Q}_p, D^0)\right) \\
&\quad \cdot \bigotimes_{\ell \notin U} \mathbf{d}_L\left(L^n \otimes^L_A \mathbf{R}\Gamma_{\mathrm{cts}}(\mathbb{Q}_\ell, V)\right)^{-1} \\
&\overset{7.2.5}{=} \mathbf{d}_L\, \mathbf{R}\Gamma_{\mathrm{cts}}(U, V_\rho) \\
&\quad \cdot \mathbf{d}_L\, \mathbf{R}\Gamma_{\mathrm{cts}}(\mathbb{Q}_p, V_\rho)^{-1} \cdot \mathbf{d}_L\, \mathbf{R}\Gamma(\mathbb{Q}_p, D^0_\rho) \\
&\quad \cdot \bigotimes_{\ell \notin U \cup \{p\}} \mathbf{d}_L\, \mathbf{R}\Gamma_{\mathrm{cts}}(\mathbb{Q}_\ell, V_\rho)^{-1}
\end{aligned}$$

$$
\begin{aligned}
&\overset{7.2.4}{\cong} \mathbf{d}_L\, \mathbf{R}\Gamma_{\mathrm{cts}}(U, V_\rho) \\
&\quad \cdot \mathbf{d}_L\, \mathbf{R}\Gamma_{\mathrm{cts}}(\mathbb{Q}_p, V_\rho)^{-1} \cdot \mathbf{d}_L\, \mathbf{R}\Gamma_f(\mathbb{Q}_p, V_\rho) \\
&\quad \cdot \bigotimes_{\ell \notin U \cup \{p\}} \left(\mathbf{d}_L\, \mathbf{R}\Gamma_{\mathrm{cts}}(\mathbb{Q}_\ell, V_\rho) / \mathbf{d}_L\, \mathbf{R}\Gamma_f(\mathbb{Q}_\ell, V_\rho)\right)^{-1} \\
&\overset{\mathrm{def}}{=} \mathbf{d}_L\, \mathbf{R}\Gamma_f(U, V_\rho) \\
&= \mathbf{1}_L.
\end{aligned}
$$

The isomorphism indicates that we introduced the acyclic complexes $\mathbf{R}\Gamma_f(\mathbb{Q}_\ell, V_\rho)$, i.e. effectively changing the isomorphism by $P_{L,\ell}(V_\rho, 1)^{\pm 1}$. Additionally, we used the quasi-isomorphisms of lemma 7.2.3(iii) and an analysis of its construction yields the following: f_{D/D^0_ρ} with $D = \mathbf{D}^\dagger_{\mathrm{rig},\mathbb{Q}_p}(V_\rho)$ is an isomorphism which creates the second quasi-isomorphism of lemma 7.2.3(iii), hence introduces the factor $\{P_{L,p}(V_\rho, u) P_{L,p}(D^0_\rho, u)^{-1}\}_{u=1}$ as $\mathbf{D}^0_{\mathrm{dR}}(D/D^0_\rho) = 0$. The first quasi-isomorphism of lemma 7.2.3(iii) can be understood as the acyclicity of $\mathbf{R}\Gamma_f(\mathbb{Q}_p, (D^0_\rho)^D)$, hence introduces the factor $P_{L,p}((D^0_\rho)^D, 1)$ as $\mathbf{D}^0_{\mathrm{dR}}((D^0_\rho)^D) = 0$. □

Corollary 7.4.5. *Let $a \in A$ and $\rho(a) \neq 0$, then*

$$\zeta_\beta(U, T, a \cdot \mathrm{lc}_{D^0})(\rho) = \det \rho(a) \cdot \zeta_\beta(U, T, \mathrm{lc}_{D^0})(\rho).$$

8 p-adic L-Functions of Motives

Hypothesis 8.0.1. Let F be a finite Galois extension over $\mathbb{Q}$, M a motive over F with an action of a finite extension of K over $\mathbb{Q}$, a place λ of K over p and a Galois extension F_∞ of F with Galois group G. We assume that G is a compact p-adic Lie group, in particular G satisfies (**) of [FK06, §1.4.2], and that only finitely many primes of F ramify in F_∞.

The motive has to also satisfy the following conditions:

(C1) Let τM be the Weil restriction of M, i.e. τM is a K-motive over $\mathbb{Q}$. Assume either a) or b) of the following:

a) The period map gives an isomorphism

$$\mathbb{R} \otimes_{\mathbb{Q}} (\tau M)_B \xrightarrow{\sim} \mathbb{C} \otimes_{\mathbb{Q}} t_{\tau M}.$$

b) F_∞ is totally real and the period map gives an isomorphism

$$\mathbb{R} \otimes_{\mathbb{Q}} (\tau M)_B^+ \xrightarrow{\sim} \mathbb{R} \otimes_{\mathbb{Q}} t_{\tau M}.$$

(C2) Let $M_\lambda = K_\lambda \otimes_{K \otimes_{\mathbb{Q}} \mathbb{Q}_p} M_p$ be the λ-adic realisation of M, which is a finite-dimensional K_λ-vector space endowed with a continuous action of $\mathrm{Gal}(\bar{F}/F)$. Then, for each place ν of F lying over p, there exists a $(\varphi, \Gamma_{F_\nu})$-direct summand $D^0_\lambda(\nu)$ over $K_\lambda \otimes \mathbf{B}^\dagger_{\mathrm{rig},F_\nu}$ of $\mathbf{D}^\dagger_{\mathrm{rig},F_\nu}(M_\lambda)$ such that

$$\mathbf{D}_{\mathrm{dR},F_\nu}(D^0_\lambda(\nu)) \cong t_{dR,\nu}(M_\lambda) := \mathbf{D}_{\mathrm{dR},F_\nu}(M_\lambda)/\mathbf{D}^0_{\mathrm{dR},F_\nu}(M_\lambda).$$

Remark 8.0.2. The condition (C2) is a (φ, Γ)-analogue of the Dabrowski-Panchishkin condition which first appeared to our knowledge in [Pot13, Prop. 3.7].

Example 8.0.3. Let $f \in \mathrm{S}_k(\Gamma_1(M), \varepsilon)$ be a normalised elliptic modular cuspidal new eigenform with $k \geq 2$. By the work of Deligne and Scholl we can attach to f a motive M_f and we consider the motive $M_f(1)$ which satisfies (C1). Assume that f has good reduction at p and that the associated p-adic representation $\mathrm{V}_p f$ has the Frobenius polynomial

$$X^2 - a_p X + \varepsilon(p) p^{k-1}.$$

Furthermore we require that the Frobenius action is semi-simple, that $\mathrm{V}_p f$ is indecomposable and that the two roots α_1 and α_2 of the Frobenius polynomial are distinct and in K. Hence, the theory described in [Pot12, §5] applies and the two (rank one) φ-eigenspaces of $\mathbf{D}_{\mathrm{crys}}(K_\lambda \otimes V_p f)(1)$ correspond to the two rank one subspaces D_{α_1} and D_{α_2} of $\mathbf{D}^\dagger_{\mathrm{rig},\mathbb{Q}_p}(K_\lambda \otimes \mathrm{V}_p f)$ which both satisfy (C2) (see also [Pot13, §3.1]).

8.1 p-adic L-Functions of Motives

Definition 8.1.1. Let $\mathcal{O}_\lambda$ be the ring of integers of K_λ, define

$$\Lambda = \mathcal{O}_\lambda[[G]]$$

and let A be some A_n in the notation of §3.5.

We fix an $\mathcal{O}_\lambda$-lattice T_0 of M_λ that is stable under the Galois action. Define

$$T^\# := \mathrm{Ind}_F^{\mathbb{Q}} \left(T_0 \otimes_{\mathcal{O}_\lambda} \Lambda^\#\right)$$

where we equip the tensor product with the diagonal Galois action. The Galois action of $\sigma \in \mathrm{Gal}(\bar{F}/F)$ on $x \in \Lambda^\#$ is defined to be $x\bar{\sigma}^{-1}$, where here $\bar{\sigma}$ denotes the canonical image of σ in $G \subset \Lambda$. We regard $\Lambda^\#$ as a (left) Λ-module via the natural left action of Λ on $\Lambda^\#$. Note that $T^\#$ is a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-module and define $A^\# = A \otimes_\Lambda \Lambda^\#$ and $V^\# = A \otimes_\Lambda T^\#$.

We also define

$$D^{0,\#} := \bigoplus_{\nu|p} \mathrm{Ind}_{F_\nu}^{\mathbb{Q}_p} \left((A \mathbin{\widehat{\otimes}} \mathbf{B}^\dagger_{\mathrm{rig},F_\nu} \otimes_{K_\lambda \otimes \mathbf{B}^\dagger_{\mathrm{rig},F_\nu}} D^0_\lambda(\nu)) \otimes_{A \mathbin{\widehat{\otimes}} \mathbf{B}^\dagger_{\mathrm{rig},F_\nu}} \mathbf{D}^\dagger_{\mathrm{rig},F_\nu}(A^\#) \right)$$

which is a $A \mathbin{\widehat{\otimes}} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}$-module.

Proposition 8.1.2. *The quadruple $(\Lambda, A, T^\#, D^{0,\#})$ fulfils hypothesis 7.0.1 assuming that there exists a primitive determinant $\Delta_A(D^{0,\#})$. Moreover $D^{0,\#}$ is even a $(\varphi, \Gamma_{\mathbb{Q}_p})$-direct summand of $\mathbf{D}^\dagger_{\mathrm{rig}}(V^\#)$.*

Proof. We just have to check 7.0.1(iii) and the direct summand claim which we verify in the following lemmas. □

Lemma 8.1.3. *$D^{0,\#}$ is a $(\varphi, \Gamma_{\mathbb{Q}_p})$-direct summand of $\mathbf{D}^\dagger_{\mathrm{rig}}(V^\#)$.*

Proof. Due to the double coset formula (see [NSW08, Prop. 1.5.11]) we have that the restriction of the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-representation $V^{\#}$ to $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ fulfils:

$$\mathrm{Res}_{\mathbb{Q}}^{\mathbb{Q}_p} V^{\#} = \mathrm{Res}_{\mathbb{Q}}^{\mathbb{Q}_p} \mathrm{Ind}_F^{\mathbb{Q}} \left(M_\lambda \otimes_{K_\lambda} A^{\#}\right) \cong \bigoplus_{\nu|p} \underbrace{\mathrm{Ind}_{F_\nu}^{\mathbb{Q}_p} \left(M_\lambda \otimes_{K_\lambda} A^{\#}\right)}_{=:V_\nu^{\#}}$$

since the set of right cosets $\mathrm{Gal}(F_\nu/\mathbb{Q}_p)\backslash\mathrm{Gal}(F/\mathbb{Q})$ for a prime $\nu|p$ of F are in bijection with the set of completions of F over p.

By lemma 5.5.1 we find

$$\begin{aligned}\mathbf{D}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}(V_\nu^{\#}) &\cong \mathbf{D}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}\left(\mathrm{Ind}_{F_\nu}^{\mathbb{Q}_p}\left(M_\lambda \otimes_{K_\lambda} A^{\#}\right)\right)\\ &\cong \mathrm{Ind}_{F_\nu}^{\mathbb{Q}_p} \mathbf{D}^{\dagger}_{\mathrm{rig},F_\nu}\left(M_\lambda \otimes_{K_\lambda} A^{\#}\right).\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\mathbf{D}^{\dagger}_{\mathrm{rig},F_\nu}\left(M_\lambda \otimes_{K_\lambda} A^{\#}\right) &\cong \mathbf{D}^{\dagger}_{\mathrm{rig},F_\nu}(M_\lambda \otimes_{K_\lambda} A) \otimes_{A\widehat{\otimes}\mathbf{B}^{\dagger}_{\mathrm{rig},F_\nu}} \mathbf{D}^{\dagger}_{\mathrm{rig},F_\nu}(A^{\#})\\ &\cong \left(A\,\widehat{\otimes}_{K_\lambda} \mathbf{D}^{\dagger}_{\mathrm{rig},F_\nu}(M_\lambda)\right) \otimes_{A\widehat{\otimes}\mathbf{B}^{\dagger}_{\mathrm{rig},F_\nu}} \mathbf{D}^{\dagger}_{\mathrm{rig},F_\nu}(A^{\#})\end{aligned}$$

where the isomorphisms hold due to lemmas 5.5.3 and 5.5.2. The statement follows. □

Lemma 8.1.4. $\mathbf{R}\Gamma(\mathbb{Q}_p, D^{0,\#})$ *is perfect which verifies 7.0.1(iii)b).*

Proof. By an analogous argument to [Pot12, Prop. 2.5(2)] (i.e. [Liu08, Thm. 4.7,Rem. 4.6], see also [KPX14, Thm. 2.3.11]) the $(\varphi, \Gamma_{F_\nu})$-module $D^0_\lambda(\nu)$ over $K_\lambda \otimes \mathbf{B}^{\dagger}_{\mathrm{rig},F_\nu}$ can be decomposed by means of short exact sequences into étale modules $\mathbf{D}^{\dagger}_{\mathrm{rig},F_\nu}(V)$, for V a $\mathrm{Gal}(\overline{F}_\nu/F_\nu)$-representation over K_λ, and twists $(K_\lambda \mathbin{\widehat{\otimes}} \mathbf{B}^{\dagger}_{\mathrm{rig},F_\nu})t^{\pm 1}$ (see remark 5.2.10).

Since $(\varphi, \Gamma_{F_\nu})$-modules are projective, these short exact sequences split, hence

$$(A \mathbin{\widehat{\otimes}} \mathbf{B}^{\dagger}_{\mathrm{rig},F_\nu} \otimes_{K_\lambda \otimes \mathbf{B}^{\dagger}_{\mathrm{rig},F_\nu}} D^0_\lambda(\nu)) \otimes_{A\widehat{\otimes}\mathbf{B}^{\dagger}_{\mathrm{rig},F_\nu}} \mathbf{D}^{\dagger}_{\mathrm{rig},F_\nu}(A^{\#})$$

can be decomposed into étale modules

$$(A \mathbin{\widehat{\otimes}} \mathbf{B}^{\dagger}_{\mathrm{rig},F_\nu} \otimes_{K_\lambda \otimes \mathbf{B}^{\dagger}_{\mathrm{rig},F_\nu}} \mathbf{D}^{\dagger}_{\mathrm{rig},F_\nu}(V)) \otimes_{A\widehat{\otimes}\mathbf{B}^{\dagger}_{\mathrm{rig},F_\nu}} \mathbf{D}^{\dagger}_{\mathrm{rig},F_\nu}(A^{\#}) \cong \mathbf{D}^{\dagger}_{\mathrm{rig},F_\nu}\left(V \otimes_A A^{\#}\right),$$

for V as above, and twists

$$(A \mathbin{\widehat{\otimes}} \mathbf{B}^{\dagger}_{\mathrm{rig},F_\nu} \otimes_{K_\lambda \otimes \mathbf{B}^{\dagger}_{\mathrm{rig},F_\nu}} (K_\lambda \mathbin{\widehat{\otimes}} \mathbf{B}^{\dagger}_{\mathrm{rig},F_\nu})t^{\pm 1})) \otimes_{A\widehat{\otimes}\mathbf{B}^{\dagger}_{\mathrm{rig},F_\nu}} \mathbf{D}^{\dagger}_{\mathrm{rig},F_\nu}(A^{\#}) \cong \mathbf{D}^{\dagger}_{\mathrm{rig},F_\nu}(A^{\#})t^{\pm 1}.$$

The statement now follows by observing that both possible ingredients have perfect cohomology by theorem 3.6.15, theorem 5.8.3 and corollary 5.7.9. Furthermore, induction leaves the cohomology groups untouched, see lemma 5.7.4. □

Regarding 7.0.1(iii)c) we have:

Lemma 8.1.5. *The class of $V^{\#,+}$ in $K_0^{(\mathrm{sh})}(A)$ and the class of $D^{0,\#}$ in $K_0^{(\mathrm{sh})}(A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p})$ both correspond to the free object of rank $r := \dim_K(\tau M)^+_B$.*

Proof. The proof of [FK06, Lem. 4.2.8] implies that the class of $\mathbf{d}_A V^{\#,+}$ in $K_0^{(\mathrm{sh})}(A)$ corresponds to $[A^r]$ for $r = \dim_K(\tau(M)^+_B)$.

Consider

$$D^{0,\#} = \bigoplus_{\nu|p} \mathrm{Ind}_{F_\nu}^{\mathbb{Q}_p} \left((A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},F_\nu} \otimes_{K_\lambda \otimes \mathbf{B}^\dagger_{\mathrm{rig},F_\nu}} D^0_\lambda(\nu)) \otimes_{A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},F_\nu}} \mathbf{D}^\dagger_{\mathrm{rig},F_\nu}(A^\#) \right).$$

Due to conjecture 6.1.6(vii), $\mathbf{D}^\dagger_{\mathrm{rig},F_\nu}(A^\#)$ is isomorphic to $A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},F_\nu}$. In particular there exists an abstract isomorphism

$$D^{0,\#} \xrightarrow{\sim} A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p} \otimes_{K_\lambda \otimes \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}} \bigoplus_{\nu|p} \mathrm{Ind}_{F_\nu}^{\mathbb{Q}_p} D^0_\lambda(\nu)$$

of $A \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}$-modules. Since $K_\lambda \otimes \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}$ is a Bézout domain (see remark 5.2.4), we deduce that $D^{0,\#}$ is free.

Furthermore, because $D^0_\lambda(\nu)$ is deRham and fulfils the Panchishkin-Dabrowski condition (C2) we see that every summand contributes

$$[F_\nu : \mathbb{Q}_p] \,\mathrm{rk}\, D^0_\lambda(\nu) = [F_\nu : \mathbb{Q}_p] \,\mathrm{rk}\, t_{dR,\nu}(M_\lambda)$$

to the rank of $D^{0,\#}$. Hence, $D^{0,\#}$ also has rank r. □

Remark 8.1.6. This implies by conjecture 6.1.6(vi) that $\Delta_A(D^{0,\#})$ and $\mathbf{d}_A V^{\#,+}$ are equal in $K_0(A)$.

Definition 8.1.7. Fix a non-empty open set $U \subseteq \operatorname{Spec} \mathbb{Z}[1/p]$ such that $F/\mathbb{Q}$, F_∞/F and M are unramified over U.

We set

$$SC(U, M, D^0_\lambda(\nu), F_\infty/F) := SC(U, V^\#, D^{0,\#})$$

and for an isomorphism

$$\beta : \tilde{A} \otimes_A \mathbf{d}_A V^{\#,+} \xrightarrow{\sim} \tilde{A} \otimes_A \Delta_{A,2}(D^{0,\#})$$

we define

$$\zeta_\beta(U, M, D^0_\lambda(\nu), F_\infty/F) \coloneqq \zeta_\beta(U, T^\#, D^{0,\#}).$$

8.2 p-adic L-Functions of Motives Independent of U

We would like to define the Selmer complex $SC(M, D^0_\lambda(\nu), F_\infty/F)$ and the p-adic L-function $\zeta_\beta(M, D^0_\lambda(\nu), F_\infty/F)$ which are independent of U.

Definition 8.2.1. For $\ell \neq p$ we define a subcomplex $C^\bullet_f(\mathbb{Q}_\ell, V)$ of $C^\bullet_{\mathrm{cts}}(\mathbb{Q}_\ell, V)$ as follows: it is 0 in degrees $\neq 0, 1$, it is $C^0(\mathbb{Q}_\ell, V)$ in degree 0 and it is the kernel of $C^1_{\mathrm{cts}}(\mathbb{Q}_\ell, V)_{\partial=0} \to H^1_{\mathrm{cts}}(\mathbb{Q}_\ell^{\mathrm{ur}}, V)$ in degree 1.

Definition 8.2.2. Let $SC(V, D^0)$ be the mapping fibre of

$$C^\bullet_{\mathrm{cts}}(U, V) \oplus C^\bullet(\mathbb{Q}_p, D^0) \oplus \bigoplus_{\ell \notin U \cup \{p\}} C^\bullet_f(\mathbb{Q}_\ell, V) \longrightarrow \bigoplus_{\ell \notin U} C^\bullet_{\mathrm{cts}}(\mathbb{Q}_\ell, V)$$

with the obvious inclusion $C^\bullet_f(\mathbb{Q}_\ell, V) \to C^\bullet_{\mathrm{cts}}(\mathbb{Q}_\ell, V)$ in the case $\ell \neq p$.

Definition 8.2.3. For each finite place ν of F not lying over p, we define $C^\bullet_f(\nu)$, a subcomplex of $C^\bullet_{\mathrm{cts}}(F_\nu, M_\lambda \otimes_{K_\lambda} A^\#)$: it is 0 in degrees $\neq 0, 1$, degree 0 is $C^0_{\mathrm{cts}}(F_\nu, M_\lambda \otimes_{K_\lambda} A^\#)$ and degree 1 is the kernel of $C^1_{\mathrm{cts}}(F_\nu, M_\lambda \otimes_{K_\lambda} A^\#)_{\partial=0} \to H^1_{\mathrm{cts}}(F_\nu^{\mathrm{ur}}, M_\lambda \otimes_{K_\lambda} A^\#)$.

Lemma 8.2.4. *We have a distinguished triangle*

$$SC(U, V^\#, D^{0,\#}) \longrightarrow SC(V^\#, D^{0,\#}) \longrightarrow \textstyle\bigoplus_\nu C^\bullet_f(\nu) \longrightarrow$$

where ν ranges over all finite places of F not lying over $U \cup \{p\}$.

Proof. For $\ell \neq p$, we have quasi-isomorphisms

$$C^\bullet_{\mathrm{cts}}(\mathbb{Q}_\ell, V^\#) \to \bigoplus_{\nu|\ell} C^\bullet_{\mathrm{cts}}(F_\nu, M_\lambda \otimes_{K_\lambda} A^\#) \qquad \text{and}$$
$$C^\bullet_f(\mathbb{Q}_\ell, V^\#) \to \bigoplus_{\nu|\ell} C^\bullet_f(\nu),$$

essentially because

$$V^{\#} \cong \bigoplus_{\nu|\ell} \mathrm{Ind}_{F_\nu}^{\mathbb{Q}_\ell} \left(M_\lambda \otimes_{K_\lambda} A^{\#} \right)$$

as seen in the proof of lemma 8.1.3. Additionally, one has to use Shapiro's lemma. □

Definition 8.2.5. We set

$$SC(M, D^0_\lambda(\nu), F_\infty/F) := SC(V^{\#}, D^{0,\#}).$$

Remark 8.2.6. In the derived category, we have canonical isomorphisms

$$C^\bullet_f(\nu) \cong C^\bullet_{\mathrm{cts}} \left(\mathrm{Gal}(F_\nu^{\mathrm{ur}}/F_\nu), (M_\lambda \otimes_{K_\lambda} A^{\#})^{I_\nu} \right) \cong \left[(M_\lambda \otimes_{K_\lambda} A^{\#})^{I_\nu} \xrightarrow{1-\varphi_\nu} (M_\lambda \otimes_{K_\lambda} A^{\#})^{I_\nu} \right].$$

Definition 8.2.7. Let Υ be the subset of the finite places of F not over p with infinite ramification index in F_∞/F.

The following proposition can be understood as the base change of [FK06, Prop. 4.2.13] along the map $\Lambda \to A$ except maybe the last statement, which however still has the same proof.

Proposition 8.2.8. *Let ν be a finite place of F not over p. Then:*

(i) The set Υ is empty if G has a commutative open subgroup.

(ii) $(M_\lambda \otimes_{K_\lambda} A^{\#})^{I_\nu}$ is projective and finitely generated.

(iii) If $\nu \in \Upsilon$, $(M_\lambda \otimes_{K_\lambda} A^{\#})^{I_\nu}$ vanishes.

(iv) Assume $\nu \notin \Upsilon$. Let A' be another Banach algebra which fulfils hypothesis 7.0.1(i) and let Y be a finitely generated projective A'-module endowed with a continuous right action of A which commutes with the action of A'. Then there is the isomorphism

$$Y \otimes^L_A (M_\lambda \otimes_{K_\lambda} A^{\#})^{I_\nu} \xrightarrow{\sim} (M_\lambda \otimes_{K_\lambda} Y^{\#})^{I_\nu}.$$

Proof. As already noted, we just have to prove (iv). By (ii) we can work on the level of complexes and just consider the normal tensor product. Then

$$\begin{aligned} Y \otimes_A (M_\lambda \otimes_{K_\lambda} A^{\#})^{I_\nu} &= Y \otimes_A (M_\lambda^{J_\nu} \otimes_{K_\lambda} A^{\#})^{I_\nu/J_\nu} \overset{(\star)}{=} \left(Y \otimes_A (M_\lambda^{J_\nu} \otimes_{K_\lambda} A^{\#}) \right)^{I_\nu/J_\nu} \\ &= (M_\lambda^{J_\nu} \otimes_{K_\lambda} Y^{\#})^{I_\nu/J_\nu} = (M_\lambda \otimes_{K_\lambda} Y^{\#})^{I_\nu} \end{aligned}$$

where $(\star)$ holds because I_ν/J_ν is finite (see the proof of [FK06, Prop. 4.2.13(2)]). □

Proposition 8.2.9. *(i) The complex $SC(M, D^0_\lambda(\nu), F_\infty/F)$ is perfect.*

(ii) $[SC(M, D^0_\lambda(\nu), F_\infty/F)] = 0$ in $K_0^{(\mathrm{sh})}(A)$.

Proof. The distinguished triangle in lemma 8.2.4 shows that $SC(M, D^0_\lambda(\nu), F_\infty/F)$ is perfect. Additionally remark 8.2.6 and corollary 7.1.3 show (ii). □

Definition 8.2.10. Let $\Sigma(M, D^0_\lambda(\nu), F_\infty/F)$ be the smallest subcategory which fulfils [FK06, §1.3.1] and contains all objects of $\mathcal{P}(A)$ which are quasi-isomorphic to the complex $SC(M, D^0_\lambda(\nu), F_\infty/F)$.

We define

$$\zeta(\nu, F_\infty/F) = \zeta(\nu, M, F_\infty/F) : \mathbf{1}_A \xrightarrow{\sim} \mathbf{d}_A\, C^\bullet_f(\nu)^{-1}$$

obtained by the obvious trivialisation of $C^\bullet_f(\nu)$ (see remark 8.2.6).

The distinguished triangle from lemma 8.2.4 implies the canonical isomorphism

$$\mathbf{d}_A\, SC(M, D^0_\lambda(\nu), F_\infty/F) \cong \mathbf{d}_A\, SC(U, M, D^0_\lambda(\nu), F_\infty/F) \cdot \bigotimes_{\substack{\nu\in S_f,\\ \nu\notin U\cup\{p\}}} \mathbf{d}_A\, C^\bullet_f(\nu).$$

Thus, the object

$$\zeta_\beta(M, D^0_\lambda(\nu), F_\infty/F) := \zeta_\beta(U, M, D^0_\lambda(\nu), F_\infty/F) \cdot \prod_{\substack{\nu\in S_f,\\ \nu\notin U\cup\{p\}}} \zeta(\nu, F_\infty/F)$$

is an element of

$$\mathrm{Isom}(\mathbf{1}_A \to \mathbf{d}_A\, SC(M, D^0_\lambda(\nu), F_\infty/F)^{-1}) \times^{K_1^{(\mathrm{sh})}(A)} K_1^{(\mathrm{sh})}(\tilde{A}).$$

We denote the class of $\zeta_\beta(M, D^0_\lambda(\nu), F_\infty/F)$ in $K_1^{(\mathrm{sh})}(A, \Sigma(M, D^0_\lambda(\nu), F_\infty/F)) \times^{K_1^{(\mathrm{sh})}(A)} K_1^{(\mathrm{sh})}(\tilde{A})$ with the same symbol.

8.3 Values of p-adic L-Functions of Motives at Twisted Artin Characters

Definition 8.3.1. Let V be a vector space over a finite extension L of $\mathbb{Q}_p$ endowed with a continuous action of $\mathrm{Gal}(\bar{F}_\nu/F_\nu)$ for ν not dividing p. Then we define the *Frobenius* *polynomial of* V to be

$$P_{L,\nu}(V,u) \coloneqq \det_L \left(1 - \varphi_\nu u | V^{I_\nu}\right)$$

where φ_ν is the geometric Frobenius. For $\nu | p$ and a $(\varphi, \Gamma_{F_\nu})$-module D over $L \otimes \mathbf{B}^\dagger_{\mathrm{rig},F_\nu}$ we define the *Frobenius polynomial of* D to be

$$P_{L,\nu}(D,u) \coloneqq \det_{L \otimes N_\nu} \left(1 - \varphi_\nu u | \mathbf{D}_{\mathrm{crys},F_\nu}(D)\right)$$

where N_ν is the maximal unramified extension in $F_\nu/\mathbb{Q}_p$ and $\varphi_\nu = \varphi^f$ with $f = [\kappa(\nu) : \mathbb{F}_p]$.

Remark 8.3.2. We think there is a typo in Fukaya-Kato's definition [FK06, §4.2.19], the determinant has to be taken over $L \otimes N_\nu$.

Lemma 8.3.3. *The rules regarding induction are:*

$$\begin{aligned} P_{L,\ell}(\mathrm{Ind}_F^{\mathbb{Q}} V, u) &= \prod_{\nu | \ell} P_{L,\ell}(\mathrm{Ind}_{F_\nu}^{\mathbb{Q}_\ell} V, u) \\ P_{L,\ell}(\mathrm{Ind}_{F_\nu}^{\mathbb{Q}_\ell} V, u) &= P_{L,\nu}(V, u^f) \\ P_{L,p}(\mathrm{Ind}_{F_\nu}^{\mathbb{Q}_\ell} D, u) &= P_{L,\nu}(D, u^f). \end{aligned}$$

Proof. The first equality follows from $\mathrm{Res}_{\mathbb{Q}}^{\mathbb{Q}_\ell} \mathrm{Ind}_F^{\mathbb{Q}}(-) \cong \bigoplus_{\nu | l} \mathrm{Ind}_{F_\nu}^{\mathbb{Q}_\ell}(-)$ as in lemma 8.1.3 and [Neu99, Prop. VII.10.4(iv)] implies the second equality with $f = [\kappa(\nu) : \mathbb{F}_\ell]$.

Regarding the third equality we have the following isomorphism

$$\mathbf{D}_{\mathrm{crys},\mathbb{Q}_p}(\mathrm{Ind}_{F_\nu}^{\mathbb{Q}_p} D) = \left(\mathbb{Z}[\Gamma_{\mathbb{Q}_p}] \otimes_{\mathbb{Z}[\Gamma_{F_\nu}]} D[1/t]\right)^{\Gamma_{\mathbb{Q}_p}} \cong D[1/t]^{\Gamma_{F_\nu}} = \mathbf{D}_{\mathrm{crys},F_\nu}(D)$$

by Shapiro's lemma with the same φ-action. Let N_ν be the maximal unramified subextension of $F_\nu/\mathbb{Q}_p$, in particular the degree $[N_\nu : \mathbb{Q}_p]$ is $f = [\kappa(\nu) : \mathbb{F}_p]$. Hence the desired statement follows by the proof of [Neu99, Prop. VII.10.4(iv)] which shows

$$\det_L \left(1 - \varphi u | \mathbf{D}_{\mathrm{crys},F_\nu}(D)\right) = \det_{L \otimes N_\nu} \left(1 - \varphi^f u^f | \mathbf{D}_{\mathrm{crys},F_\nu}(D)\right).$$

□

Hypothesis 8.3.4. Let $(M, F_\infty/F, (D^0_\lambda(\nu))_\nu)$ fulfil hypothesis 8.0.1. For $j \in \mathbb{Z}$ assume one of the following is fulfilled:

(i) $j = 0$.

(ii) We are in the situation a) of condition (C1), F_∞ contains $\mathbb{Q}(\zeta_{p^\infty})$ and $M^j_{dR} = M^0_{dR}$.

(iii) We are in the situation b) of condition (C1), F_∞ contains $\mathbb{Q}(\zeta_{p^\infty})^+$, j is even and $M^j_{dR} = M^0_{dR}$.

Lemma 8.3.5. *Assuming hypothesis 8.3.4, the triple $(M(j), F_\infty/F, (D^0_\lambda(\nu)(j))_\nu)$ also fulfils hypothesis 8.0.1.*

Proof. The condition on the deRham realisation ensures the invariance of the tangent spaces under the twist. Hence (C2) is still true for the new triple. In the cases (i) and (ii) condition (C1) remains true. In case (iii) the parity of j ensures that the twist leaves the real part of the Betti realisation unchanged, hence (C1) always follows. □

Proposition 8.3.6. *Let j be as in hypothesis 8.3.4. Let $\chi^{-j} : G \to \mathbb{Z}_p^\times$ be the homomorphism induced by $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_p^\times : \sigma \mapsto \chi(\sigma)^{-j}$ which is the $(-j)$-th power of the cyclotomic character.*

Let K' be a finite extension of K, $\rho : G \to GL_n(K')$ be a homomorphism which factors through $\mathrm{Gal}(F_m/F)$ for some m, $[\rho^*]$ be the associated Artin K'-motive over F and let $M(\rho^*)$ be the K'-motive $[\rho^*] \otimes_K M$ over F. Fix a place λ' of K' lying over λ and let $L = (K')_{\lambda'}$.

We assume that the map

$$\rho\chi^{-j} : \Lambda \to GL_n(K')$$

factors through an A_m with $m < \infty$. Note that $\rho\chi^{-j}$ factors automatically through A_m for $m \gg 0$.

Furthermore, let

$$V = M(\rho^*)(j)_{\lambda'} = [\rho^*]_{\lambda'} \otimes_{K_\lambda} M_\lambda(j),$$
$$\tau(V) = \mathrm{Ind}_F^{\mathbb{Q}} V.$$

Assume the following:

(i) $H^0(\mathbb{Q}, \tau(V))$, $H^1_f(\mathbb{Q}, \tau(V))$, $H^0(\mathbb{Q}, \tau(V)^*(1))$, $H^1_f(\mathbb{Q}, \tau(V)^*(1))$ *all vanish.*

(ii) $P_{L,\nu}(V, 1) \neq 1$ *if $\nu \in \Upsilon$ in case 1, respectively for all finite places ν of F not lying over U or p in case 2.*

(iii) For any place ν of F lying over p, let

$$D^0(\nu) := \mathbf{D}^{\dagger}_{\mathrm{rig},F_\nu}([\rho^*]_{\lambda'}(j)) \otimes_{K_\lambda \otimes \mathbf{B}^{\dagger}_{\mathrm{rig},F_\nu}} D^0_\lambda(\nu).$$

Then $P_{L,\nu}(V,u)P_{L,\nu}(D^0(\nu),u)^{-1}$ and $P_{L,\nu}(D^0(\nu)^D,u)$ do not have a zero at $u=1$.

Then $L^n \otimes^L_A SC(M, D^0_\lambda(\nu), F_\infty/F)$ (case 1) and $L^n \otimes^L_A SC(U, M, D^0_\lambda(\nu), F_\infty/F)$ (case 2) are acyclic where A acts on L^n from the right via $\rho\chi^{-j}$ (see definition 7.2.1).

Proof. We have

$$V_{\rho\chi^{-j}} \stackrel{\mathrm{def}}{=} L^n \otimes_A V^{\#} = \mathrm{Ind}_F^{\mathbb{Q}} \left(L^n \otimes_A (M_\lambda \otimes_{K_\lambda} A^{\#})\right) = \tau(V)$$

which can be easily checked by working out the Galois action on the modules. Furthermore,

$$\begin{aligned}
D^0_\rho &= L^n \otimes_A D^{0,\#} \\
&= \bigoplus_{\nu|p} \mathrm{Ind}_{F_\nu}^{\mathbb{Q}_p} \left(L^n \otimes_A \left(\mathbf{D}^{\dagger}_{\mathrm{rig},F_\nu}(A^{\#}) \otimes_{A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},F_\nu}} A \,\widehat{\otimes}_{K_\lambda} D^0_\lambda(\nu)\right)\right) \\
&= \bigoplus_{\nu|p} \mathrm{Ind}_{F_\nu}^{\mathbb{Q}_p} \left(\mathbf{D}^{\dagger}_{\mathrm{rig},F_\nu}([\rho^*]_{\lambda'}(j)) \otimes_{A \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},F_\nu}} A \,\widehat{\otimes}_{K_\lambda} D^0_\lambda(\nu)\right) \\
&= \bigoplus_{\nu|p} \mathrm{Ind}_{F_\nu}^{\mathbb{Q}_p} \left(\mathbf{D}^{\dagger}_{\mathrm{rig},F_\nu}([\rho^*]_{\lambda'}(j)) \otimes_{K_\lambda \otimes \mathbf{B}^{\dagger}_{\mathrm{rig},F_\nu}} D^0_\lambda(\nu)\right) \\
&= \bigoplus_{\nu|p} \mathrm{Ind}_{F_\nu}^{\mathbb{Q}_p} D^0(\nu)
\end{aligned}$$

because $\mathbf{D}^{\dagger}_{\mathrm{rig}}$ commutes with base change (see theorem 5.5.2).

Hence, case 2 follows from corollary 7.2.5 when we use lemma 8.3.3 to relate the Frobenius polynomials. A similar computation shows that one does not need $P_{L,\nu}(V,1) \neq 1$ unless $\nu \in \Upsilon$ in case 1. □

Theorem 8.3.7. *We again assume the validity of the ζ- and ϵ-isomorphism conjectures as in theorem 7.4.4.*

(i) Analogously to theorem 7.4.4, under the corresponding boundary map, the element $\zeta_\beta(U, M, D^0_\lambda(\nu), F_\infty/F)$ maps to the class of $SC(U, M, D^0_\lambda(\nu), F_\infty/F)$ and $\zeta_\beta(M, D^0_\lambda(\nu), F_\infty/F)$ maps to the class of $SC(M, D^0_\lambda(\nu), F_\infty/F)$.

(ii) Make assumptions as in proposition 8.3.6: then $L_{K'}(M(\rho^*), s) = L_{K'}(\tau(M(\rho^*)), s)$

has no pole at $s = j$ and the value of $\zeta_\beta(M, D^0_\lambda(\nu), F_\infty/F)$ at $\rho\chi^{-j}$ is

$$\left\{\frac{L_{K'}(M(\rho^*), j)}{\Omega_\infty(\tau(M(\rho^*))(j))}\right\} \cdot \Omega_p(\tau(M(\rho^*))(j)) \cdot \prod_{r\geq 1} \Gamma(r)^{h(j-r)} \\ \cdot \prod_{\nu|p}\left(\left\{\frac{P_{L,\nu}(V,u)}{P_{L,\nu}(D^0(\nu),u)}\right\}_{u=1} \cdot P_{L,\nu}(D^0(\nu)^D, 1)\right) \cdot \prod_{\nu\in\Upsilon} P_{L,\nu}(V,1)$$

and the value of $\zeta_\beta(U, M, D^0_\lambda(\nu), F_\infty/F)$ at $\rho\chi^{-j}$ is

$$\left\{\frac{L_{K'}(M(\rho^*), j)}{\Omega_\infty(\tau(M(\rho^*))(j))}\right\} \cdot \Omega_p(\tau(M(\rho^*))(j)) \cdot \prod_{r\geq 1} \Gamma(r)^{h(j-r)} \\ \cdot \prod_{\nu|p}\left(\left\{\frac{P_{L,\nu}(V,u)}{P_{L,\nu}(D^0(\nu),u)}\right\}_{u=1} \cdot P_{L,\nu}(D^0(\nu)^D, 1)\right) \cdot \prod_{\nu|\ell\notin U\cup\{p\}} P_{L,\nu}(V,1).$$

Proof. This is just theorem 7.4.4 and observing the identities found in lemma 8.3.3. □

8.4 Refined Formula for Values of p-adic L-Functions of Motives over $\mathbb{Q}$ at Twisted Artin Characters

Assume that $F = \mathbb{Q}$ and that we are in case 8.0.1(C1)b) (resp. a)). Furthermore we assume that all the φ-eigenvalues of $\mathbf{D}_{\mathrm{crys}}(D^0_\lambda(p))$ are contained in K_λ, this implies that $D^0_\lambda(p)$ is triangulable (see [Ber11, Thm. 3.3.4/p.6]). As in [FK06, §4.2.24] let γ^+ be a K-basis of M_B^+ (resp. γ^- a K-basis of M_B^-) and let δ be a K-basis of t_M. These bases can be used to define periods

$$\Omega(\gamma^+, \delta) \in \mathbb{C}^\times \qquad (\text{resp. } \Omega(\gamma^-, \delta) \in \mathbb{C}^\times)$$

for a fixed embedding $K \to \mathbb{C}$, see [FK06, §4.2.24(1)].

Fukaya-Kato also define an isomorphism $\mathbf{d}_A\, A^r \cong \mathbf{d}_A\, V^{\#,+}$ using γ^+ (resp. γ). We would now like to define an isomorphism

$$\tilde{A} \otimes_A \mathbf{d}_A\, A^r \xrightarrow{\sim} \tilde{A} \otimes_A \Delta_{A,2}(D^{0,\#}).$$

Note that

$$\begin{aligned}\tilde{A}\otimes_A \Delta_{A,2}(D^{0,\#}) &= \tilde{A}\otimes_A \Delta_{A,2}\big((A\,\widehat{\otimes}\,\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}\otimes_{K_\lambda\otimes\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}} D^0_\lambda(p))\otimes_{A\,\widehat{\otimes}\,\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}} \mathbf{D}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}(A^{\#})\big)\\ &\cong \tilde{A}\otimes_A \Delta_{A,2}\big(A\,\widehat{\otimes}\,\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}\otimes_{K_\lambda\otimes\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbb{Q}_p}} D^0_\lambda(p)\big)\\ &\cong \tilde{A}\otimes_{K_\lambda}\Delta_{K_\lambda,2}(D^0_\lambda(p))\end{aligned}$$

where the first isomorphism is due to conjecture 6.1.6 parts (viii) and (vii) and second isomorphism is according to part (i).

Now we use

$$\Delta_{K_\lambda,2}(D^0_\lambda(p)) \xrightarrow{\theta_{dR,L,\xi}(D^0_\lambda(p))} \mathbf{d}_{K_\lambda}\,\mathbf{D}_{\mathrm{dR}}(D^0_\lambda(p)) \xrightarrow{\sim} \mathbf{d}_{K_\lambda}(K_\lambda\otimes t_M) \xrightarrow{\sim} \mathbf{d}_{K_\lambda}(K^r_\lambda)$$

where the first isomorphism is defined in [Nak13, p. 38] and the last one is induced by δ. Hence by base change we can now define our isomorphism $\tilde{A}\otimes_A \mathbf{d}_A\, A^r \cong \tilde{A}\otimes_A \Delta_{A,2}(D^{0,\#})$.

Composing Fukaya-Kato's isomorphism and our isomorphism yields the isomorphism

$$\beta_{\gamma,\delta}:\quad \tilde{A}\otimes_A \mathbf{d}_A\, V^{\#,+} \xrightarrow{\sim} \tilde{A}\otimes_A \mathbf{d}_A\, A^r \xrightarrow{\sim} \tilde{A}\otimes_A \Delta_{A,2}(D^{0,\#}).$$

The above construction allows us to deduce:

Theorem 8.4.1. *We again assume the validity of the ζ- and ϵ-isomorphism conjectures as in theorem 7.4.4.*

Assume the setting of proposition 8.3.6. Additionally assume $F=\mathbb{Q}$ and $\beta=\beta_{\gamma,\delta}$ as above. Then the value of $\zeta_\beta(M, D^0_\lambda(\nu), F_\infty/F)$ at $\rho\chi^{-j}$ is

$$\left\{\frac{L_{K'}(M(\rho^*),j)}{(2\pi i)^{nj\,\mathrm{rk}_K t_M}\cdot\Omega(\gamma^+,\delta)^{d(\rho,j,+)}\cdot\Omega(\gamma^-,\delta)^{d(\rho,j,-)}}\right\}\cdot\epsilon(\rho^*,\xi)^{-d}\cdot(p^j\nu^{-1})^{f_p(\rho)}$$
$$\cdot\prod_{r\geq 1}\Gamma(r)^{h(j-r)}\cdot\left(\left\{\frac{P_{L,p}(V,u)}{P_{L,p}(D^0(p),u)}\right\}_{u=1}\cdot P_{L,p}(D^0(p)^D,1)\right)\cdot\prod_{\ell\in\Upsilon}P_{L,\ell}(V,1).$$

and the value of $\zeta_\beta(U,M,D^0_\lambda(\nu),F_\infty/F)$ at $\rho\chi^{-j}$ is

$$\left\{\frac{L_{K'}(M(\rho^*),j)}{(2\pi i)^{nj\,\mathrm{rk}_K t_M}\cdot\Omega(\gamma^+,\delta)^{d(\rho,j,+)}\cdot\Omega(\gamma^-,\delta)^{d(\rho,j,-)}}\right\}\cdot\epsilon(\rho^*,\xi)^{-d}\cdot(p^j\nu^{-1})^{f_p(\rho)}$$
$$\cdot\prod_{r\geq 1}\Gamma(r)^{h(j-r)}\cdot\left(\left\{\frac{P_{L,p}(V,u)}{P_{L,p}(D^0(p),u)}\right\}_{u=1}\cdot P_{L,p}(D^0(p)^D,1)\right)\cdot\prod_{\ell\notin U\cup\{p\}}P_{L,\ell}(V,1)$$

where

(i) $d(\rho, j, +)$ *(resp.* $d(\rho, j, -)$*) denotes the* K'*-dimension of the part of the Betti realisation* $[\rho]$ *on which complex conjugation acts as multiplication by* $(-1)^j$ *(respectively* $(-1)^{j-1}$*) (in the case 8.0.1(C1)b),* $\Omega(\gamma^-, \delta)^{d(\rho,j,-)}$ *is defined to be* 1*),*

(ii) $\epsilon(\rho^*, \xi)$ *is the* ϵ*-constant associated to* ρ^* *in the sense of [FK06, §3.2.2],*

(iii) $\nu = \det_{K_\lambda}(\varphi | \mathbf{D}_{\mathrm{crys}}(D^0_\lambda(p)))$*,*

(iv) $d = \dim_K t_M$*, and*

(v) $f_p(\rho)$ *is the* p*-adic order of the Artin conductor of* ρ*.*

Proof. Verbatim copy of the proof of [FK06, Thm. 4.2.26(1)]. $\square$

We now specialise the previous formula to the case of modular forms:

Example 8.4.2. Assume we are in the situation of example 8.0.3, for $F_\infty = \mathbb{Q}_p(\mu_{p^\infty})$. Then $\zeta_\beta(M_f(1), D_{\alpha_i}, \mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ at $\rho\chi^{-j}$ for $0 \le j \le k-2$ evaluates to

$$\frac{1}{\alpha_i^{f_p(\rho)}} \cdot \left(1 - \frac{\rho(p)\varepsilon(p)p^{k-j-2}}{\alpha_i}\right) \cdot \left(1 - \frac{\rho^*(p)p^j}{\alpha_i}\right) \cdot \frac{j! p^{(j+1)f_p(\rho)}}{\epsilon(\rho^*, \xi)} \cdot \left\{\frac{L_{K'}(\mathrm{V}_p f(\rho^*), j+1)}{(2\pi i)^j \cdot \Omega(\gamma^\pm, \delta)}\right\}$$

where $\epsilon(\rho^*, \xi)$ is a Gauß sum associated with ρ^* with respect to ξ as defined in [FK06, 3.2.2(7)][1] and the sign is determined by $(-1)^j \rho(-1)$.

Proof. These j fulfil 8.3.4(ii). We have $n = 1$, $\mathrm{rk}_K t_M = 1$, $d = 1$, $\Upsilon = \varnothing$, $h(j-r) = 1$ if and only if $j + 1 - r = 0, k-1$ and $\nu = p^{-1} \cdot \alpha_i$. The action of the complex conjugation on the Betti realisation is given by $\rho(-1)$, hence the sign formula follows from the definition of $d(\rho, j, \pm)$. Furthermore

$$\begin{aligned}
\left\{\frac{P_{L,p}(V, u)}{P_{L,p}(D^0_{\alpha_i}(p), u)}\right\}_{u=1} &= P_{L,p}(D^0_{\alpha_{3-i}}(p), 1) \\
&= 1 - \det_{K'}\left(\varphi \,\middle|\, \mathbf{D}_{\mathrm{crys}}\left(\mathbf{D}^\dagger_{\mathrm{rig},\mathbb{Q}_p}([\rho^*]_{\lambda'}(j)) \otimes_{K_\lambda \otimes \mathbf{B}^\dagger_{\mathrm{rig},\mathbb{Q}_p}} D^0_{\alpha_{3-i}}(p)\right)\right) \\
&= 1 - \rho(p) \cdot p^{-j} \cdot p^{-1} \cdot \alpha_{3-i} \\
&= 1 - \frac{\rho(p) \cdot p^{-j-1} \cdot \varepsilon(p) \cdot p^{k-1}}{\alpha_i} \\
&= 1 - \frac{\rho(p) \cdot \varepsilon(p) \cdot p^{k-j-2}}{\alpha_i}
\end{aligned}$$

[1]Note that the defintion is not the standard Gauß sum definition, see also [LVZ15, footnote 1, p.4] for a small correction.

since $\alpha_1 \cdot \alpha_2 = \varepsilon(p) \cdot p^{k-1}$ and

$$\begin{aligned} P_{L,p}(D^0_{\alpha_i}(p)^D, 1) &= 1 - p^{-1} \cdot (\rho(p) \cdot p^{-j} \cdot p^{-1} \cdot \alpha_i)^{-1} \\ &= 1 - \frac{\rho(p)^{-1} \cdot p^j}{\alpha_i}. \end{aligned}$$

□

9 p-adic L-Functions of Motives over the Cyclotomic Extension

Hypothesis 9.0.1. We still assume hypothesis 8.0.1 in this section. Additionally we require that the field F_∞ is a finite extension of the cyclotomic $\mathbb{Z}_p$-extension of F and that the *abelian* Galois group G can be decomposed as $\mathbb{Z}_p \times \Delta$, where Δ is a finite group.

Then $\Lambda(G)$ can written as $\Lambda(\mathbb{Z}_p) \otimes \Lambda(\Delta)$. Recall that χ is the cyclotomic character and let γ be a topological generator of $\mathbb{Z}_p$.

9.1 The Algebra $A_\infty(\mathbb{Z}_p)$

The algebra $A_\infty(\mathbb{Z}_p)$ was studied by Lazard (see [Laz62]) and he proved the following properties.

Theorem 9.1.1 ([Laz62, (7.3)]). *The algebra $A_\infty(\mathbb{Z}_p)$ is a Bézout domain.*

Furthermore we have the following relationship between divisibility and zero behaviour:

Lemma 9.1.2 ([Laz62, (4.7)]). *For $f, g \in A_\infty(\mathbb{Z}_p)$ the following holds:*

$$f|g \quad \Leftrightarrow \quad (f) \leq (g),$$

where (h) is the principal divisor of $h \in A_\infty(\mathbb{Z}_p)$ (see [Laz62, (4.6)]).

The above choice of γ enables us to identify $A_\infty(\mathbb{Z}_p)$ with a subset of power series in $\mathbb{C}_p[[X]]$ which converge on the open unit disc.

Definition 9.1.3. For power series $f(X), g(X) \in \mathbb{C}_p[[X]]$ which converge on the open unit disc of $\mathbb{C}_p$ we define

$$|f(X)|_r := \sup_{|z|_p < r} |f(z)|_p$$

and we say $f(X) = O(g(X))$ (resp. $f(X) = o(g(X))$) if

$$|f(X)|_r = O(|g(X)|_r) \text{ for } r \to 1^-$$
$$(\text{resp. } |f(X)|_r = o(|g(X)|_r) \text{ for } r \to 1^-).$$

Furthermore we denote the property $f(X) = O(g(X))$ and $g(X) = O(f(X))$ by $f(X) \sim g(X)$.

We now define special elements in $A_\infty(\mathbb{Z}_p)$.

Definition 9.1.4. For $q \geq 0$ let μ_{p^q} be the set of primitive p^q-th roots of unity in $\bar{\mathbb{Q}}_p$ and let $\mu_{p^\infty} = \bigcup \mu_{p^q}$ be the group of p-power roots of unity in $\bar{\mathbb{Q}}_p$.

Definition 9.1.5. We fix once and for all $\varphi(p^q)$-th roots of p ($q \geq 1$) which we denote by $\sqrt[\varphi(p^q)]{p}$, where φ is Euler's totient function.

Lemma 9.1.6. *Assume that S is a subset of $\{j_1, \dots, j_2\} \times (\mu_{p^\infty} \setminus \{1\})$ for $j_1, j_2 \in \mathbb{Z}$ such that for all j and almost all $q \geq 1$ the set $S \cap \{j\} \times \mu_{p^q}$ is either empty or equal to $\{j\} \times \mu_{p^q}$. Let $q(S) < \infty$ be the largest q for which the previous statement does not hold.*

Set

$$\log_{p,S} \coloneqq \prod_{(j,\zeta) \in S} \frac{(\chi(\gamma)^{-j} \cdot \gamma - \zeta)}{\sqrt[\varphi(p^{q(\zeta)})]{p}}$$

where $q(\zeta)$ is chosen such that $\zeta \in \mu_{p^{q(\zeta)}}$.

Then the product converges to an element in $A_\infty(\mathbb{Z}_p)$ where we used as K the finite extension $\mathbb{Q}_p(\mu_{p^{q(S)}}, \sqrt[\varphi(p^{q(S)})]{p})$ of $\mathbb{Q}_p$. The zeroes are precisely of the form $\chi(\gamma)^j \cdot \zeta$ for $(j, \zeta) \in S$ and all these zeroes are simple.

Proof. The statement follows from the proof of [Pol03, Lem. 4.1] when we notice that

$$\prod_{\zeta \in \mu_{p^q}} \frac{(\chi(\gamma)^{-j} \cdot \gamma - \zeta)}{\sqrt[\varphi(p^q)]{p}} = \frac{\Phi_{p^q}(\chi(\gamma)^{-j} \cdot \gamma)}{p}$$

since $\#\mu_{p^q} = \varphi(p^q)$ where $\Phi_{p^q}(X)$ is the p^q-th cyclotomic polynomial. □

Remark 9.1.7. We say that a character ρ of $\mathbb{Z}_p$ corresponds to an element in S if $\rho = \chi^j \cdot \rho_\zeta$ for $(j, \zeta) \in S$, where ρ_ζ is the character of $\mathbb{Z}_p$ which sends γ to ζ. We might also write ρ 'in' S.

Definition 9.1.8. We define the *p-adic logarithm* to be

$$\log_p \coloneqq (\gamma - 1) \cdot \log_{p,\{0\}\times(\mu_{p^\infty}\setminus\{1\})}.$$

Remark 9.1.9. Note that $\log_p \sim \log_{p,\{0\}\times(\mu_{p^\infty}\setminus\{1\})}$.

Lemma 9.1.10. *Let S be given by congruence conditions, i.e. for $j \in \{j_1, \dots, j_2\}$ we are given a set $C_j \subset \mathbb{Z}/m_j\mathbb{Z}$ and*

$$S = \left\{ (j,\zeta) \,\middle|\, \begin{array}{l} j_1 \le j \le j_2, \\ q \bmod c_j \in C_j, \\ \zeta \in \mu_{p^q} \end{array} \right\}.$$

Let $d(S) \coloneqq \sum_j \#C_J/m_j$ be the natural density of S. Then

$$\log_{p,S} \sim (\log_{p,\{0\}\times(\mu_{p^\infty}\setminus\{1\})})^{d(S)} \sim (\log_p)^{d(S)}.$$

Proof. For $p^{-\frac{1}{p^{N-1}(p-1)}} \le r \le p^{-\frac{1}{p^{(N+1)-1}(p-1)}}$ we have

$$\begin{aligned} |\log_S|_r &= \prod_{\substack{q \bmod c_j \in C_j \\ q \le N+1}} p \cdot r^{p^{q-1}(p-1)} \\ &= p^{\sum_{q \bmod c_j \in C_j, q \le N+1} 1} \cdot r^{\sum_{q \bmod c_j \in C_j, q \le N+1} p^{q-1}(p-1)} \end{aligned}$$

by [Spr12, Ex. 4.2]. Hence

$$\begin{aligned} \frac{|\log_S|_r}{(|\log_{p,\{0\}\times(\mu_{p^\infty}\setminus\{1\})}|_r)^{d(S)}} = {} & p^{\sum_{q \bmod c_j \in C_j, q \le N+1} 1 - d(S)\cdot(N+1)} \\ & \cdot r^{(p-1)\cdot\left(\sum_{q \bmod c_j \in C_j, q \le N+1} p^{q-1} - d(S) \sum_{q \le N+1} p^{q-1}\right)}. \end{aligned}$$

Since S is given by congruence conditions, the difference $\sum_{q \bmod c_j \in C_j, q \le N+1} 1 - d(S)\cdot(N+1)$ is bounded, hence the first factor is bounded by positive constants from above and from below. For the same reason we get that

$$\begin{aligned}\sum_{q \bmod c_j \in C_j, q \leq N} p^{q-1} &= \frac{1}{p} \sum_j \sum_{[c_j] \in C_j} \sum_{c_j + m_j \cdot n \leq N} p^{c_j + m_j \cdot n} \\ &= \frac{1}{p} \sum_j \sum_{[c_j] \in C_j} p^{c_j} \sum_{n \leq (N-c_j)/m_j} p^{n m_j} \\ &= \sum_j \sum_{[c_j] \in C_j} p^{c_j - 1} \frac{p^{m_j(\lfloor (N-c_j)/m_j \rfloor + 1)} - 1}{p-1}\end{aligned}$$

where the c_j are the lifts of elements in C_j such that $0 \leq c_j < m_j$. Therefore the exponent of the second factor is

$$\sum_j \sum_{[c_j] \in C_j} p^{c_j} \cdot (p^{m_j(\lfloor (N+1-c_j)/m_j \rfloor + 1)} - 1) - d(S) \cdot (j_2 - j_1 + 1) \cdot (p^{N+2} - 1).$$

It suffices to check the endpoints of the interval $[p^{-\frac{1}{p^{N-1}(p-1)}}, p^{-\frac{1}{p^{(N+1)-1}(p-1)}}]$. Hence the second factor can also be bounded from above and from below by positive constants. $\square$

Remark 9.1.11. Changing S by finitely many elements does not affect the growth behaviour since the extra factors are bounded (see [Spr12, Ex. 4.2]).

Lemma 9.1.12. *The greatest common divisor of* $\log_{p,S_1}$ *and* $\log_{p,S_2}$ *is* $\log_{p,S_1 \cap S_2}$.

Proof. It is clear that $\log_{p,S_1 \cap S_2} | \log_{p,S_i}$ by lemma 9.1.2.

Assume that $f | \log_{p,S_i}$ for both i, then the principal divisors fulfil $(f) \leq (\log_{p,S_i})$. Hence $(f) \leq (\log_{p,S_1 \cap S_2})$, i.e. $f | \log_{p,S_1 \cap S_2}$. $\square$

9.2 Associated Power Series

Remark 9.2.1. We will also use the modified p-adic L-function $\zeta_\beta(M, D^0_\lambda(\nu), a, F_\infty/F)$ where the local condition morphism lc_{D^0} is multiplied with $a \in A_\infty(G)$. Fix such an a for this section. We obviously have $\zeta_\beta(M, D^0_\lambda(\nu), F_\infty/F) = \zeta_\beta(M, D^0_\lambda(\nu), 1, F_\infty/F)$.

Remark 9.2.2. Since characters of finite groups are linearly independent, for all $n \leq \infty$ the canonical map

$$\begin{aligned} A_n(G) &\longrightarrow A_n(\mathbb{Z}_p)^{\hat{\Delta}} \\ a &\longmapsto (\hat{\delta}(a))_{\hat{\delta} \in \hat{\Delta}} \end{aligned}$$

is an isomorphism of $A_n(\mathbb{Z}_p)$-modules.

Definition 9.2.3. Let ι be the involution $\sigma \to \sigma^{-1}$ for $\sigma \in G$ which extends to an involution ι of A_∞.

Definition 9.2.4. For $\zeta \in \mu_{p^\infty}$ we let ρ_ζ be the character of $\mathbb{Z}_p$ which sends γ to ζ.

Remark 9.2.5. The character ρ_ζ as above has conductor p^{q+1} if $\zeta \in \mu_{p^q}$.

Hypothesis 9.2.6. Assume that $j_1, \dots, j_2$ fulfil 8.3.4(ii). Furthermore we require that the L-functions $L_{K'}(M(\rho), j)$ do not have a pole for $j_1 \leq j \leq j_2$ and for all $\hat{\delta} \in \hat{\Delta}$ the sets

$$S^{\hat{\delta}}_{M,a} := \left\{ (j,\zeta) \;\middle|\; \begin{array}{l} -j_2 \leq j \leq -j_1, \\ \zeta \in \mu_{p^\infty} \setminus \{1\}, \\ \hat{\delta}\rho_\zeta\chi^j \text{ fulfils the assumptions of case 2 in prop. 8.3.6}, \\ a(\hat{\delta}\rho_\zeta\chi^j) \neq 0 \end{array} \right\}$$

fulfil the hypothesis of lemma 9.1.6.

Definition 9.2.7. Set

$$\begin{aligned} A_{n,\log,S^{\hat{\delta}}_{M,a}} &:= A_n(\mathbb{Z}_p)/(\log_{p,S^{\hat{\delta}}_{M,a}}) \qquad \text{for } n \leq \infty, \\ SC(M, D^0_\lambda(\nu), a, F_\infty/F)_{\log,S^{\hat{\delta}}_{M,a}} &:= A_{\infty,\log,S^{\hat{\delta}}_{M,a}} \otimes_{A_\infty(G)} SC(M, D^0_\lambda(\nu), a, F_\infty/F) \qquad \text{and} \\ \zeta_\beta(M, D^0_\lambda(\nu), a, F_\infty/F)_{\log,S^{\hat{\delta}}_{M,a}} &:= A_{\infty,\log,S^{\hat{\delta}}_{M,a}} \otimes_{A_\infty(G)} \zeta_\beta(M, D^0_\lambda(\nu), a, F_\infty/F) \end{aligned}$$

where the map $A_\infty(G) \to A_\infty(\mathbb{Z}_p)$ is induced by $\hat{\delta} \in \hat{\Delta}$.

Remark 9.2.8. Sometimes we will drop $(M, D^0_\lambda(\nu), a, F_\infty/F)$ to shorten the notation, i.e. we will write $\zeta_{\log,S^{\hat{\delta}}_{M,a}}$ instead of $\zeta_\beta(M, D^0_\lambda(\nu), a, F_\infty/F)_{\log,S^{\hat{\delta}}_{M,a}}$.

Proposition 9.2.9. *The complex $SC(M, D^0_\lambda(\nu), a, F_\infty/F)_{\log,S^{\hat{\delta}}_{M,a}}$ is locally, i.e. over $A_{n,\log,S^{\hat{\delta}}_{M,a}}$ for every $n < \infty$, acyclic.*

Proof. We first note that $A_{n,\log,S^{\hat{\delta}}_{M,a}}$ is a (commutative) affinoid, i.e. the quotient by a maximal ideal $\mathfrak{m}$ is a finite field extension $K_\mathfrak{m}$ of $\mathbb{Q}_p$ hence a maximal ideal corresponds to a map

$$\rho : \mathbb{Z}_p \to K_\mathfrak{m}$$

which fulfils $\rho(a+b) = \rho(a) \cdot \rho(b)$ and $\rho(0) = 1$. Hence $\rho(\gamma)$ cannot be zero, where γ as before is a topological generator of $\mathbb{Z}_p$. Thus $\rho(\gamma)$ is invertible, i.e. $\rho : \mathbb{Z}_p \to K^\times_\mathfrak{m}$ is a character, and $\rho(\gamma)$ has to be a zero of $\log_{p,S^{\hat{\delta}}_{M,a}}$ otherwise ρ does not factor through $A_{n,\log,S^{\hat{\delta}}_{M,a}}$. Hence, ρ corresponds to an element in $S^{\hat{\delta}}_{M,a}$ (see lemma 9.1.6).

Since these characters fulfil the assumptions of case 2 in proposition 8.3.6 by definition, we can deduce from this proposition 8.3.6 that for all maximal ideals $\mathfrak{m}$ of $A_{n,\log,S^{\hat{\delta}}_{M,a}}$, the complex

$$A_{n,\log,S^{\hat{\delta}}_{M,a}}/\mathfrak{m} \otimes_{A_{\infty,\log,S^{\hat{\delta}}_{M,a}}} SC(M, D^0_\lambda(\nu), a, F_\infty/F)_{\log,S^{\hat{\delta}}_{M,a}}$$

is acyclic, hence we deduce by [KPX14, Lem. 4.1.5] that

$$A_{n,\log,S^{\hat{\delta}}_{M,a}} \otimes_{A_{\infty,\log,S^{\hat{\delta}}_{M,a}}} SC(M, D^0_\lambda(\nu), a, F_\infty/F)_{\log,S^{\hat{\delta}}_{M,a}}$$

is acyclic. □

Definition 9.2.10. Since $SC(M, D^0_\lambda(\nu), a, F_\infty/F)_{\log,S^{\hat{\delta}}_{M,a}}$ is locally acyclic, there is the following canonical map

$$A_{\infty,\log,S^{\hat{\delta}}_{M,a}} \otimes_{A_\infty} K_1^{\mathrm{sh}}(A_\infty, \Sigma(U, M, D^0_\lambda(\nu), a, F_\infty/F)) \to \varprojlim K_1(A_{n,\log,S^{\hat{\delta}}_{M,a}}).$$

Hence consider the following maps

$$\begin{aligned} \varprojlim K_1(A_{n,\log,S^{\hat{\delta}}_{M,a}}) \times^{K_1^{\mathrm{sh}}(A_n)} K_1^{\mathrm{sh}}(\tilde{A}_n) &\xrightarrow{\det} \varprojlim \tilde{A}^\times_{n,\log,S^{\hat{\delta}}_{M,a}} \cong \tilde{A}^\times_{\infty,\log,S^{\hat{\delta}}_{M,a}} \subset \tilde{A}_{\infty,\log,S^{\hat{\delta}}_{M,a}} \\ \zeta_\beta(M, D^0_\lambda(\nu), a, F_\infty/F)_{\log,S^{\hat{\delta}}_{M,a}} &\longmapsto \zeta'_{\log,S^{\hat{\delta}}_{M,a}} \end{aligned}$$

Set $S_{a,b} := \{a, \ldots, b\} \times (\mu_{p^\infty} \setminus \{1\})$. Then $\log_{p,S_{-j_2,-j_1} \setminus S^{\hat{\delta}}_{M,a}}$ is coprime to $\log_{p,S^{\hat{\delta}}_{M,a}}$ by lemma 9.1.12, hence the Chinese remainder theorem (together with the Bézout property) induces

$$\begin{array}{ccccc} \tilde{A}_{\infty,\log,S^{\hat{\delta}}_{M,a}} & \longrightarrow & \tilde{A}_{\infty,\log,S^{\hat{\delta}}_{M,a}} \oplus \tilde{A}_{\infty,\log,S_{-j_2,-j_1} \setminus S^{\hat{\delta}}_{M,a}} & \xrightarrow{\sim} & \tilde{A}_{\infty,\log,S_{-j_2,-j_1}} \\ \zeta'_{\log,S^{\hat{\delta}}_{M,a}} & \longmapsto & (\zeta'_{\log,S^{\hat{\delta}}_{M,a}}, 0) & \longmapsto & \zeta^{\hat{\delta}}_\beta(M, D^0_\lambda(\nu), a, F_\infty/F)_{\log}. \end{array}$$

Define

$$\zeta_\beta(M, D^0_\lambda(\nu), a, F_\infty/F)_{\log} := \iota\left((\zeta^{\hat{\delta}}_\beta(M, D^0_\lambda(\nu), a, F_\infty/F)_{\log})_{\hat{\delta}}\right) \in A_\infty(G)/(\log_{p,S_{j_1,j_2}})$$

using remark 9.2.2 and the involution ι defined in 9.2.3.

Lemma 9.2.11. *Assume that $f \in A_\infty$ has the same values as $\zeta_\beta(M, D^0_\lambda(\nu), a, F_\infty/F)_{\log}$ at the characters $\hat{\delta}\rho\chi^j$, where $\hat{\delta} \in \hat{\Delta}$, ρ is a finite order character and $j_1 \leq j \leq j_2$. In other words, if $\rho^*\chi^{-j}$ corresponds to an element in $S^{\hat{\delta}^{-1}}_{M,a}$, then its value should be given by theorem 8.3.7, otherwise we require the evaluation to be 0. Then f is a lift of $\zeta_\beta(M, D^0_\lambda(\nu), a, F_\infty/F)_{\log}$.*

Proof. Let $g \in A_\infty$ be a lift of $\zeta_\beta(M, D^0_\lambda(\nu), a, F_\infty/F)_{\log}$. By assumption $f - g$ has zeroes at all characters corresponding to elements in S_{j_1,j_2}. Hence $f - g$ is divisible by $\log_{p,S_{j_1,j_2}}$ (see lemmas 9.1.2 and 9.1.6). □

A slight generalisation of [Spr12, Lem. 6.11]:

Lemma 9.2.12. *There can be at most one lift $f \in A_\infty$ of $\zeta_\beta(M, D^0_\lambda(\nu), a, F_\infty/F)_{\log}$ with the property $f = O(\log_p^h)$ such that $h < j_2 - j_1 + 1$.*

Proof. Assume that there are two distinct functions f and g with these properties. We have that $f - g \neq 0$ has zeroes at the characters corresponding to S_{j_1,j_2}, hence $f - g$ is divisible by $\log_{p,S_{j_1,j_2}}$. We conclude

$$\log_{p,S_{j_1,j_2}} = O(f - g) \subseteq O(\log_p^h)$$

which contradicts $\log_{p,S_{j_1,j_2}} \sim \log_p^{j_2-j_1+1}$ (see the discussion leading up to [Pol03, Prop. 2.11]). □

Definition 9.2.13. The unique lift (if it exists) as in the last lemma we will call the *least $\log_p$-growth lift of* $\zeta_\beta(M, D^0_\lambda(\nu), a, F_\infty/F)$ and will be denoted by $\zeta_\beta(M, D^0_\lambda(\nu), a, F_\infty/F)_\infty$.

In the special situation of modular forms one should suspect that the above theory connects to known results and indeed the unique least $\log_p$-growth lift is just the Mazur-Tate-Teitelbaum p-adic L-function:

Example 9.2.14. Assume the situation of example 8.0.3, i.e. that our motive comes from a modular form f. We think of $\mathbb{Q}(f)$ and $\mathbb{Q}(\alpha)$, where α is a root of the Frobenius polynomial, as fields embedded in $\bar{\mathbb{Q}}$. Also fix embeddings $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Set $K := \mathbb{Q}(f, \alpha, \mu_{p-1})$.

Since we want to compare two constructions we have to fix the free parameters in both constructions. In Mazur-Tate-Teitelbaum's p-adic L-function we set the periods $\Omega_f^\pm$ to be $\pm\Omega(\gamma^\pm, \delta)$, where $\Omega(\gamma^\pm, \delta)$ was defined in section 8.4. Furthermore, $\zeta_\beta(M_f(1), D_\alpha, \mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ depends on the choice of a basis of $\mathbb{Z}_p(1)$. We let ξ be the compatible system of unit roots $(\exp(-2\pi i/n))_n$, considered as a system in $\bar{\mathbb{Q}}_p$ via the above embeddings. Note that due to our choices the Gauß sum $\epsilon(\rho, \xi^{-1})$ of a character ρ of $\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ considered as an element of $\mathbb{C}$ via the above embeddings coincides with the usual definition of the Gauß sum $\tau(\rho^*)$ of the character ρ^* (see [FK06, 3.2.2(7)]). In particular we see

$$\epsilon(\rho, \xi) = \overline{\epsilon(\rho^*, \xi^{-1})} = \overline{\tau(\rho)} = \rho(-1)\tau(\rho^*).$$

Then Mazur-Tate-Teitelbaum construct the p-adic L-function $L_p^{\mathrm{MTT}}(f, \alpha) \in A_\infty$ associated to f in [MTT86, §13] where A_∞ was formed using K_λ (see [Pol03, Def. 2.9]).

We now relate the values of the two p-adic L-functions:

$$\begin{aligned} L_p^{\mathrm{MTT}}(f, \alpha)(\rho\chi^j) &= \frac{1}{\alpha^{f_p(\rho)}} \cdot \left(1 - \frac{\rho^*(p)\varepsilon(p)p^{k-j-2}}{\alpha}\right) \cdot \left(1 - \frac{\rho(p)p^j}{\alpha}\right) \\ &\qquad \cdot \frac{j!p^{(j+1)f_p(\rho)}}{\tau(\rho^*)} \cdot \left\{\frac{L_{K'}(f_{\rho^*}, j+1)}{(-2\pi i)^j \cdot \Omega_f^\pm}\right\} \\ &= \frac{1}{\alpha^{f_p(\rho)}} \cdot \left(1 - \frac{\rho^*(p)\varepsilon(p)p^{k-j-2}}{\alpha}\right) \cdot \left(1 - \frac{\rho(p)p^j}{\alpha}\right) \\ &\qquad \cdot \frac{j!p^{(j+1)f_p(\rho)}}{\rho(-1)\epsilon(\rho, \xi)} \cdot \left\{\frac{L_{K'}(\mathrm{V}_p f(\rho), j+1)}{(-1)^j 2\pi i \cdot (-1)^j \rho(-1)\Omega(\gamma^\pm, \delta)}\right\} \\ &= \zeta_\beta(M_f(1), D_\alpha, \mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)(\rho^*\chi^{-j}) \end{aligned}$$

where the first equality follows from [Pol03, Prop. 2.11], the second equality follows from the above choices and the last one is the content of example 8.4.2.

The above calculation shows that $L_p^{\mathrm{MTT}}(f, \alpha)$ is a lift of $\zeta_\beta(M, D_\lambda^0(\nu), a, F_\infty/F)_{\log}$ due to the involution ι used in definition 9.2.10.

Furthermore, $0 \leq j \leq k-2$ fulfil 8.3.4(ii), i.e. $j_1 = 0$ and $j_2 = k-2$. The Mazur-Tate-Teitelbaum p-adic L-function $L_p^{\mathrm{MTT}}(f, \alpha)$ is contained in $O(\log_p^{\mathrm{ord}_p(\alpha)})$. Assume that $\mathrm{ord}_p(\alpha) < k-1 = j_2 - j_1 + 1$, e.g. this is the case in the supersingular case, then we deduce that the lift is unique, i.e.

$$\zeta_\beta(M_f(1), D_\alpha, \mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)_\infty = L_p^{\mathrm{MTT}}(f, \alpha) \in A_\infty.$$

In more generality, using the above results as a guideline, we conjecture the following or something similar to be true:

Remark 9.2.15. Assume that the range of j_1 and j_2 is chosen to be maximal, $a = 1$ and in the notation of theorem 8.3.7 let $h_{\max}$ be the maximal value of $\nu_p(\nu)$ for the different

ν corresponding to each j. Then we suspect that there exists a lift $f \sim \log_p^{h_f}$ with $h_f \leq h_{\max} - 1$. In particular if $h_{\max} - 1 < j_2 - j_1 + 1$ it follows that there should exist a unique least $\log_p$-growth lift $\zeta_\beta(M, D^0_\lambda(\nu), a, F_\infty/F)_\infty$.

It would be very interesting to test this idea using more examples.

9.3 Pollack's $\pm$-Construction for Modular Forms with $a_p = 0$

We now assume that we are in the setting of example 8.0.3, i.e. we are dealing with a motive of a modular form and we additionally assume that $a_p = 0$, i.e. $\alpha_2 = -\alpha_1$.

Definition 9.3.1. Set

$$\begin{aligned} S_j &:= \{0, \dots, j\} \times (\mu_{p^\infty} \setminus \{1\}), & \log_{p,j} &:= \log_{p,S_j}, \\ S_j^+ &:= \{0, \dots, j\} \times \bigcup_{n \in \mathbb{Z}^+} \mu_{p^{2n}}, & \log^+_{p,j} &:= \log_{p,S_j^+}, \\ S_j^- &:= \{0, \dots, j\} \times \bigcup_{n \in \mathbb{Z}^+} \mu_{p^{2n-1}} & \text{and } \log^-_{p,j} &:= \log_{p,S_j^-}. \end{aligned}$$

Definition 9.3.2. $\log^+_{p,j}$ and $\log^-_{p,j}$ are coprime, hence by theorem 9.1.1 there exist β^+ and β^- in A_∞ with the Bézout property

$$\log^+_{p,k-2} \cdot \beta^+ + \log^-_{p,k-2} \cdot \beta^- = 1.$$

Definition 9.3.3. We define

$$\begin{aligned} \zeta(f, \alpha_i) &:= \zeta_\beta(M_f(1), D_{\alpha_i}, F_\infty/F), \\ \zeta^\pm_G(f, \alpha_i) &:= \zeta_\beta(M_f(1), D_{\alpha_i}, \log^\pm_{p,k-2} \cdot \beta^\pm, F_\infty/F), \\ \zeta^\pm(f, \alpha_i) &:= \zeta_\beta(M_f(1), D_{\alpha_i}, \beta^\pm, F_\infty/F) \end{aligned}$$

and we set $\zeta^\pm_?(f, \alpha_i)(\rho) = 0$ if ρ is a zero of $\log^\pm_{p,k-2} \cdot \beta^\pm$ or $\beta^\pm$ respectively.

Remark 9.3.4. We immediately deduce from the Bézout property and corollary 7.4.5 that as functions with values in $\bar{\mathbb{Q}}_p$

$$\zeta^+_G(f, \alpha_i) + \zeta^-_G(f, \alpha_i) = \zeta_\beta(f, \alpha_i)$$

holds.

Remark 9.3.5. Let ρ be a character corresponding to S^+_{k-2}. Then this is a zero of $\log^+_{p,k-2}$, hence $\log^-_{p,k-2}(\rho) \cdot \beta^-(\rho) = 1$. Similarly for ρ a character corresponding to S^-_{k-2}, we have $\log^+_{p,k-2}(\rho) \cdot \beta^+(\rho) = 1$.

In particular we can deduce

$$\zeta^+_\beta(f, \alpha_i)(\rho) = \begin{cases} 0 & \text{for } \rho \text{ 'in' } S^+_{k-2} \\ \zeta_\beta(f, \alpha_i)(\rho) & \text{for } \rho \text{ 'in' } S^-_{k-2} \cap S_{M_f(1)} \end{cases}$$

and

$$\zeta^-_\beta(f, \alpha_i)(\rho) = \begin{cases} \zeta_\beta(f, \alpha_i)(\rho) & \text{for } \rho \text{ 'in' } S^+_{k-2} \cap S_{M_f(1)} \\ 0 & \text{for } \rho \text{ 'in' } S^-_{k-2}. \end{cases}$$

Remark 9.3.6. Analogously to Pollack [Pol03] we also find by evaluating the values of the p-adic L-functions (see example 8.4.2 and remark 9.2.5) that $\zeta^+_G(f, \alpha_i)_{\log}$ and $\zeta^-_G(f, \alpha_i)_{\log}/\alpha_i$ are independent of α_i. Furthermore lemma 9.2.11 and example 9.2.14 imply that

$$G^+ := \frac{L_p^{\mathrm{MTT}}(f, \alpha_1) + L_p^{\mathrm{MTT}}(f, \alpha_2)}{2}$$

is the unique least growth lift of $\zeta^+_G(f, \alpha_i)_{\log}$ and

$$\alpha_i \cdot G^- := \alpha_i \cdot \frac{L_p^{\mathrm{MTT}}(f, \alpha_1) - L_p^{\mathrm{MTT}}(f, \alpha_2)}{2\alpha_1}$$

is the unique least growth lift of $\zeta^-_G(f, \alpha_i)_{\log}$. Note that both G^+ and G^- are independent of α_i.

Remark 9.3.7. Looking at the zero behaviour we find that $G^\pm$ is divisible by $\log^\pm_{p,k-2}$. Set

$$L_p^\pm(f) := \frac{G^\pm}{\log^\pm_{p,k-2}} \in A_\infty$$

and we know that $L_p^\pm(f) \in O(\log_p^0)$. Hence $L_p^\pm(f)$ is the unique least growth lift of $\zeta^\pm(f, \alpha_i)_{\log, S_{M_f(1)} \cap S^\mp_{k-2}}$.

In particular Pollack's decomposition [Pol03, Thm. 5.1]

$$L_p(f, \alpha_i) = \log^+_{p,k-2} \cdot L_p^+(f) + \alpha_i \cdot \log^-_{p,k-2} \cdot L_p^-(f)$$

holds.

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